Skolem's Paradox	Instructor: Susan / Paddy

Lecture 1: Model Theory

Week 1 of 1	*Mathcamp 2010*

In this pair of lectures, we're going to prove that we can find a model of ZFC within a countable universe; in other words, we're going to show how we can encapsulate (amongst other things) the entire ordinal hierarchy $\omega_0, \omega_1, \ldots$ within the universe $\mathbb{N}$.

Wait, what?

1 Formalism!

Definition. A **first-order language** consists of the following:

- $\mathcal{F}$, a collection of function symbols,
- $\mathcal{R}$, a collection of relation symbols, and
- *arity*, a map from $\mathcal{F} \cup \mathcal{R} \to \mathbb{N}$ that assigns an arity[1] to each function and relation.

Example. The language of arithmetic:

$$\mathcal{L} := \{0, S, +, \cdot, <\},$$

where 0 is a 0-ary function, S, the successor function, is a unary function, $+$ and $\cdot$ are binary functions, and $<$ is a binary relation.

The language of posets:

$$\mathcal{L} := \{\leq\},$$

where $\leq$ is a binary relation (typically denoting order.)

The language of graphs:

$$\mathcal{L} := \{\sim\},$$

where $\sim$ is a binary relation (typically denoting that an edge exists between the two things it relates.)

The word "language" suggests that we might want to create things like "sentences" out of these languages. The following definitions make this concept precise:

Definition. Fix some countable set Var of variables. Then, for a language $\mathcal{L}$, we define a **term** t as follows:

- x is a term, for any $x \in Var$.
- If $t_1 \ldots t_n$ are terms and f is a n-ary function in $\mathcal{F}$, then $f(t_1, \ldots t_n)$ is a term.

[1] the **arity** of a function or relation is just the number of arguments it takes in.

Example. In the language of arithmetic, the following expressions are terms:

$$S(S(S(S(S(0))))), \quad (x \cdot x) + (y \cdot y), \quad (x + y) + z) \cdot 0.$$

What do these typically represent?

Definition. For a fixed language $\mathcal{L}$ and a countable set Var of variables, we define an **atomic formula** f as an expression of the form

$$R(t_1, \dots t_n),$$

where R is a n-ary relation in $\mathcal{R} \cup \{=\}$. ($=$ is a binary relation which we'll throw into any language, because it's kind of necessary if we're going to talk about mathematics.)

Example. In the language of arithmetic, the following expressions are atomic formulae:

$$S(0) < S(S(0)), \quad x = x \cdot x, \quad (x + y) \cdot z = (x \cdot z) + (y \cdot z).$$

What do these formulas usually denote in mathematics?

Definition. For a fixed language $\mathcal{L}$ and a countable set Var of variables, we recursively define a **well-formed formula** f as one of the following expressions:

- f, where f is an atomic formula
- $(\neg A)$, $(A \wedge B)$, $(A \vee B)$, $(A \Rightarrow B)$, $(A \Leftrightarrow B)$, where A, B are well-formed formulae.
- $(\exists x_i A), (\forall x_i A)$, where A is a well-formed formula.

Example. In the language of arithmetic, the following expressions are well-formed formulas:

$\forall x \exists y (x \leq y)$
$\forall x \forall y \forall z \left(((x < y) \wedge (y < z)) \Rightarrow (x < z)\right),$
$\forall x \exists y ((x = 0) \vee (x \cdot y = S(0))),$
$\forall x \exists y_1 \exists y_2 \exists y_3 ((x = y_1) \vee (x = y_2) \vee (x = y_3)),$
$\forall x \exists y \exists z ((\exists a (x = ((S(S(0)) \cdot a) + S(0)) \vee (\forall b \forall c ((b \cdot c = y) \vee (b \cdot c = z) \Rightarrow (b = 1 \vee c = 1)) \wedge x = y + z).$

Translate these sentences into standard English. What are they representing?

So: in the above sentences, we've been asking for "interpretations" of these sentences. How can we make this idea explicit?

2 Structures

Definition. A **structure** M for a language $\mathcal{L}$ is the following:

- A nonempty set M, which we call the **universe** of the structure
- For every n-ary function element $f \in \mathcal{F}$, a function $f^M : M^n \to M$.
- For every n-ary relation element $R \in \mathcal{R}$, a relation $R^M \subset M^n$.

In other words, M is just a way of interpreting this language $\mathcal{L}$ – i.e. a way of assigning meaning to $\mathcal{L}$'s symbols (which, intrinsically, don't have any meaning on their own.)

Example. Consider the following different interpretations of the language of arithmetic:

1. Consider the integers $\mathbb{Z}$ as our universe, where we interpret
   - 0 as the constant function that returns 0,
   - S as the unary function that returns $x + 1$ on input x,
   - $+$ as the binary function that takes in two integers and returns their sum,
   - $\cdot$ as the binary function that takes in two integers and returns their product, and
   - $<$ as the binary relation that evaluates (x, y) to true iff x is less than y.
2. Consider the real numbers $\mathbb{R}$ as our universe, where we interpret everything just as above: i.e. interpret
   - 0 as the constant function that returns 0,
   - S as the unary function that returns $x + 1$ on input x,
   - $+$ as the binary function that takes in two real numbers and returns their sum,
   - $\cdot$ as the binary function that takes in two real numbers and returns their product, and
   - $<$ as the binary relation that evaluates (x, y) to true iff x is less than y.
3. Alternately, consider again $\mathbb{Z}$ as our universe, where we interpret
   - 0 as the constant function that returns 0,
   - S as the unary function that returns $x - 1$ on input x,
   - $+$ as the binary function that takes in two integers and returns their sum,
   - $\cdot$ as the binary function that takes in two integers and returns their product, and
   - $<$ as the binary relation that evaluates (x, y) to true iff x is greater than y.

   This looks much like our first structure, except our ordering is backwards! – i.e. we think that $-2 > -1 > 0 > 1 > \ldots$ in this model.

   - Finally, consider the following rather "stupid" model of the language of arithmetic, again set in the universe of $\mathbb{Z}$:

- 0 as the constant function that returns 0,
- S as the unary function that returns 0 on input x,
- $+$ as the binary function that takes in two integers and returns 0,
- $\cdot$ as the binary function that takes in two integers and returns 0, and
- $<$ as the binary relation that never returns true.

This is indeed a structure! Just, um, not remotely what we'd like.

So: as demonstrated above, structures can be rather perverse and strange things. For example, it's possible to build things in the language of arithmetic that don't look at all like what we'd like! How can we fix this?

Well: given a structure S, we can now interpret various well-formed formulas in that structure! I.e. consider again the structure $S = \langle \mathbb{Z}, 0, S, +, \cdot, < \rangle$, where we interpret everything normally (i.e. we work in the integers with addition/multiplication/sucessor/zero/orderings all as normal.)

Then, the sentence

$$\varphi = \forall x \exists y (x \leq y)$$

is just the claim that for any x, there's always a larger y in our universe! This is clearly true in our structure: to denote this, we write

$$S \models \varphi.$$

Conversely, consider the sentence

$$\phi = \forall x \forall y (\neg(x = y)) \vee (\exists z (x < z < y) \vee (y < z < x)).$$

This says that for any two distinct elements of our structure, that there is a third element between them; a statement that is clearly false for the integers (i.e $x = 1, y = 2$). In this case, we write

$$S \not\models \varphi.$$

So: sentences give us a way to talk about what things are "true" and what things are "false" for a given structure! In specific, consider the following definition:

Definition. For a structure S, define the theory of S, $Th(S)$, to be the collection of all well-formed formulas φ such that $S \models \varphi$.

Earlier, when we studied just languages, we saw that it was really easy to "misinterpret" a language: i.e. there were lots of ways in which the symbols we wrote down could mean something that we didn't.

Theories give us a way to get around this stumbling block! How? Well: suppose you want to talk abstractly about some specific group: i.e. take for example the group $\langle \mathbb{Z}_5, 0, + \rangle$, the cyclic group of order 6, and consider it as a structure. What's its theory?

Well, enumerating everything in its theory is impossible, if only for the trivial reasons that if ϕ is a well-formed formula that $\mathbb{Z}_6$ satisfies, then $\exists y\phi$ is such a formula; thus, by nesting $\exists$ we can get infinitely many formulas[2] However, we can see that the following formulas are true of $\mathbb{Z}_6$:

$$\begin{aligned}
M &\models \forall x\forall y\exists z(x+y=z)\\
&\models \forall x\exists y(x+y=0)\\
&\models \forall x\forall y\forall z((x+y)+z=x+(y+z))\\
&\models \forall x(x+0)=x\\
&\models \forall x\forall y(x+y=y+x)\\
&\models \forall y\exists y_1\ldots\exists y_6\left(\bigwedge_{i\neq j}\neg(x_i=x_j)\right)\wedge\left(\bigvee_{i=1}^{6}(y=x_i)\right)
\end{aligned}$$

How many structures satisfy these sentences? Well, the first 5 are precisely the axioms for an abelian group, and the sixth specifies that the set has precisely 6 elements in it: so any structure that satisfies these sentences must in fact be $\mathbb{Z}_6$, up to isomorphism!

Thus, if we want to abstractly capture the idea of what it means to "be" something, a theory seems like a good idea! In other words, if two objects satisfy the same set of well-formed formulas, then it seems like they must have a high degree of similarity.

Must they always be the same, though? To answer this question, consider the structure $S = \langle\mathbb{N},<\rangle$, the natural numbers under their typical ordering. This structure satisfies certain formulas, like

$$\begin{aligned}
S &\models \forall x\exists y(x<y)\\
&\models \exists y\forall x(y<x)\vee(y=x)\\
&\models \forall x\exists y\forall z(x<y)\wedge((z<x)\vee(z=x)\vee(z=y)\vee(y<z))\\
&\models \forall x\forall y\forall z((x<y)\wedge(y<z))\Rightarrow(x<z)\\
&\models \forall x\forall y((x<y)\vee(y<x)\vee(x=y))\\
&\models \forall x\forall y\neg((x<y)\wedge(y<x))\\
&\models \forall x\neg(x<x).
\end{aligned}$$

However, these formulas all satisfy the structure $\langle\mathbb{R},<_{wo}\rangle$ with respect to some well-ordering $<_{wo}$ of $\mathbb{R}$ – and, as it turns out, there's no way[3] to create a well-formed-formula that allows us to distinguish between the two!

So, even if we have two structures that satisfy all of the same sentences, they might still be different. A natural question, then, is to consider all of mathematics! – in other

[2]There is a way of talking about a theory having a finite **axiomatization**, in which finitely many formulas can be found that imply everything that's true in the theory: but that's not where we're headed in this class.

[3]If you'd like to see why, come talk to me and I can point you towards some papers!

words, look at the structure that's given by the model of mathematics given by ZFC, the Zermelo-Fraenkel set theory axioms + the Axiom of Choice. Are there different structures of ZFC? Are there new ways of interpreting the basic language of mathematics – new ways of creating the ordinals, real number line, and such things – that are fundamentally different? Is there, as we claimed at the start of lecture, *really* a way of creating a structure with universe $\mathbb{N}$ that models all of ZFC?

Tune in next time to find out!

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Homework 1: Model Theory!, the HW

Week 4 *Mathcamp 2010*

1. Write a sentence φ in the language $\mathcal{L} = \{R\}$, where R is a binary relation, that says that a structure $M \models \varphi$ iff M has an even number of elements in its universe (or M has an infinite universe.)

2. Write a sentence φ in the language $\mathcal{L} = \{R, f\}$, where R is a binary relation and f is a binary function, that says that a structure $M \models \varphi$ iff M has n^2-many elements in its universe, for some n (or M has an infinite universe.)

3. Write a sentence φ in the language $\mathcal{L} = \{R\}$, where R is a binary relation, that says that a structure $M \models \varphi$ iff R is an equivalence relation on M.

4. Suppose that S is a structure for the language $\{<\}$ that satisfies the following three sentences:

$$\forall x \exists y (x < y)$$
$$\forall x \forall y \forall z ((x < y) \wedge (y < z) \Rightarrow (x < z))$$
$$\forall x \neg (x < x)$$

   What can you say about the size of S?

5. Let $S = \langle \mathbb{N}, < \rangle$ be the normal structure of the natural numbers under the ordering $<$, and let $T = \langle \mathbb{N}, <_T \rangle$ be the ordering on $\mathbb{N}$ defined as follows:

   $(n <_t m)$ holds iff $(n, m$ are both even or both odd, and $n < m)$ or $(n$ is even and m is odd.)

   Find a sentence φ in the language $\{<\}$ such that S satisfies φ,, but T does not.