The Unit Distance Graph	Instructor: Paddy
Homework 2: Rational Colorings	
Week 4	*Mathcamp 2010*

The first question on this set deals with our material today; the rest consists of justifying our claim, made yesterday, that our König-tree results yesterday allow us to say that the chromatic number of the plane is just the maximal chromatic number of the collection of finite graphs of Euclidean dimension 2. It's fun! Especially if you like set theory.

1. Using the ideas in class, can you bound the chromatic number (in ZFC) of $\mathbb{Q}(\sqrt{2})$[1]? How about $\mathbb{Q}(\sqrt{k})$?

2. Zorn's lemma, in mathematics, says the following:

   Lemma 1 (Zorn's lemma) *Suppose that $(P, \leq)$ is a partially ordered set[2] with the following property: in any totally ordered subset[3] $T \subset P$, there is an upper bound u: i.e. an element u such that for any other $a \in T$, $a \leq t$.*

   Then P has a maximal element: i.e. there is some $m \in P$ such that we never have $m < a$ for any $a \in P$.

   Prove this, using the axiom of choice[4]

3. How is Zorn's lemma related to König's lemma? In other words: in what ways are they similar? In what ways are they distinct? Is there any reasonable way in which you could argue that one is "stronger" than another?

4. Define property P for graphs as follows: a graph G satisfies property P iff all of its finite subgraphs are k-colorable.

   Use Zorn's lemma to prove the following claim: If V is some set of vertices and S is the collection of all graphs on V that satisfy P, then there is a graph $G \in S$ such that adding any edge to G removes it from S.

5. Prove that in such a graph G, the relation of nonadjacency amongst vertices is an equivalence relation. Conclude that the complement graph $\bar{G}$ of G is made up of a disjoint union of complete graphs.

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[1] $\mathbb{Q}(\sqrt{2})$is the ring formed by all numbers of the form $a + b\sqrt{2}, a, b \in \mathbb{Q}$.

[2] A partially ordered set $(P, \leq)$ is a set P and a binary relation $\leq$ such that $\leq$ satisfies the folowing three axioms:

- $a \leq a$ (reflexivity)
- $a \leq b$ and $b \leq a$ implies $a = b$ (antisymmetry)
- $a \leq b$ and $b \leq c$ implies $a \leq c$ (transitivity).

[3] A subset T of P is called totally ordered if every two elements are comparable in T: i.e. for any $a, b \in T$, $a \leq b$ or $b \leq a$.

[4] The Axiom of Choice says the following: for every collection of sets $\{S_i\}_{i \in I}$, there is a function f such that for every $i \in I$, $f(i) \in S_i$.

6. Show that there are no more than k components in $\bar{G}$. Conclude that G is k-colorable.