The Unit Distance Graph	Instructor: Paddy

Homework 3: $\chi(\mathbb{R}^2)$ and the Axiom of Choice

Week 4	*Mathcamp 2010*

1. Show that the well-ordering principle is equivalent to the axiom of choice. (If you didn't do it yesterday, an easy way to do this is to show that the axiom of choice implies Zorn's lemma, which proves the well-ordering principle, which proves the axiom of choice.)

2. Mimicking our proof in class today, consider the graph G on $\mathbb{R}^2$, with edges between points (x, y), (a, b) iff their difference $(x - a, y - b)$ is of the form $(q_1, q_2 + \sqrt{2})$ or $(q_1 + \sqrt{2}, q_2)$. Show that $\chi_{ZFC}(G) = 2$, while $\chi_{ZFS}(G) > \aleph_0$.

3. Let Σ be a collection of subsets of $\mathbb{R}$ closed under taking countable unions and complements. We say that a function $\mu : \Sigma \to \mathbb{R} \cup \{\pm\infty\}$ is a **measure** on Σ iff it satisfies the following properties:

   (a) (Non-negativity): $\mu(E) \geq 0$, for every element $E \in \Sigma$.

   (b) (Null empty set): $\mu(\emptyset) = 0$.

   (c) (Countable additivity): $\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i)$.

   Suppose that we are working in ZFC. Show that if μ is a measure on $\mathcal{P}(\mathbb{R})$, the collection of all subsets of $\mathbb{R}$, then μ cannot be translation-invariant[1]. In other words, show that if μ is a translation-invariant measure, then it cannot be defined on all of $\mathbb{R}$; consequently, there are non-measurable sets!

[1] A measure on the real number line is called translation-invariant if for any subset $S \subset \mathbb{R}, r \in \mathbb{R}$, we have that $\mu(S) = \mu(S + r)$, where $S + r := \{x + r : x \in S\}$.