The Unit Distance Graph	Instructor: Paddy
Lecture 2: Coloring $\mathbb{Q}^n$	
Week 1 of 1	*Mathcamp 2010*

1 Glossary

In these definitions, n denotes a natural number, G is some abelian group, h is an element of G, and S is a subset of G.

n-coloring A n-coloring of an abelian group G is just a partition of G's elements into n different sets.

h-alternating A n-coloring of G is said to be h-alternating iff for every $g \in G$, the elements

$$g, g+h, g+h+h = g+2h, \dots g+(n-1)h$$

are all different colors. (by kh, where $k \in \mathbb{Z}$ and $h \in G$, we mean the element of G denoted by adding k copies of h together.)

S-alternating A n-coloring of G is said to be S-alternating iff it's h-alternating for every $h \in S$.

weakly n-free A subset $S \subset G$ is called weakly n-free iff for any collection $\{m_h\}_{h \in S}$ of integers indexed by the elements of S, with only finitely many elements not equal to 0, we have the following implication:

$$\left(\sum_{h\in S} m_h \cdot h = 0\right) \quad \Rightarrow \quad \left(\sum_{h\in S} m_h \equiv 0 \mod n\right)$$

2 Coloring $\mathbb{Q}^2$

Theorem 1 *If S is weakly n-free, then there is a S-alternating n-coloring of G.*

Proof. Let H be the subgroup generated by S. Color H by dividing it into subsets $B_1, \dots B_n$ defined as follows:

$$B_k = \left\{ \sum_{h\in S} m_h \cdot h \,\middle|\, \sum_{h\in S} m_h \equiv k \mod n \right\}$$

Because S is weakly n-free, we know that these sets partition H. So: do the same thing to all of H's cosets! This generates a n-coloring of G that's S-alternating, by construction; so we're done!

Theorem 2 *If there is a S-alternating 2-coloring of G, then S is weakly 2-free.*

Proof. So: a S-alternating 2-coloring is just a partition of G into two sets B_1, B_2 so that for any $g \in G, h \in S$, exactly one of $\{g, g+h\}$ lives in B_1 and the other lives in B_2. Consequently, we have that for any $b \in B_i, h \in S$, $b + mh \in B_i$ iff m is even!

So: specifically consider the identity element 0. Suppose that $0 \in B_i$. Then, we know that $0 + m_h h = m_h h \in B_i$ iff m_h is even; more generally, we know that in fact

$$\sum_{h \in S} m_h h \in B_1 \text{ iff } \sum_{h \in S} m_h \text{ is even,}$$

by considering parity arguments. But this is exactly the definition for weakly 2-free!

Theorem 3 *We have the following results for the chromatic numbers of rational spaces:*

$$\chi(\mathbb{Q}^2) = 2, \chi(\mathbb{Q}^3) = 2, \chi(\mathbb{Q}^4) > 2.$$

Proof. So: by our earlier work, it suffices to show that

$$S = \{(x, y) \in \mathbb{Q} | x^2 + y^2 = 1, x = 1 \text{ or } y > 0\}$$

is weakly 2-free, as this will give us a S-alternating 2-coloring of $\mathbb{Q}$ – i.e. a partition of $\mathbb{Q}^2$ into two parts B_1, B_2 such that if $x \in B_1$, no points that are distance 1 from x are also in B_1!

So: look at solutions of $x^2 + y^2 = 1$ in $(\mathbb{Q}^+)^2$: these are in fact pairs of numbers of the form $(a/c, b/c)$ where (a, b, c) is a primitive Pythagorean triple. Consequently, we always have that exactly 1 of a, b are odd, one is even, and c is odd.

So: think of S as something of the form $\{(1, 0), (0, 1)\} \cup \{(a_i, b_i)\}_{i=1}^{\infty}$, and examine any possible sum of the form

$$n(1, 0) + r(0, 1) + \sum_{i=1}^{\infty} m_i(a_i/c_i, b_i/c_i) = (0, 0)$$

where all but finitely many of the m_i are zero. Then, we have that specifically

$$n \sum_{i=1}^{\infty} m_i \cdot a_i/c_i = 0$$

and

$$r + \sum_{i=1}^{\infty} m_i \cdot b_i/c_i = 0.$$

So: let c be the product of all of the c_i where m_i is nonzero. This is a finite odd number (b/c all of the c_i's are odd; thus, if we multiply through by 2, we have

$$n \sum_{i=1}^{\infty} m_i a_i \equiv 0 \mod 2$$

and

$$r + \sum_{i=1}^{\infty} m_i \cdot b_i \equiv 0 \mod 2.$$

Adding these together, we have that

$$\begin{aligned} & n + r + \sum_{i=1}^{\infty} m_i a_i + \sum_{i=1}^{\infty} m_i \cdot b_i \equiv 0 \mod 2 \\ \Rightarrow & n + r + \sum_{i=1}^{\infty} m_i (a_i + b_i) \equiv 0 \mod 2. \end{aligned}$$

But in any pythagorean triple (a, b, c), $a + b$ is odd! So we have in fact that

$$n + r + \sum_{i=1}^{\infty} m_i \equiv 0 \mod 2;$$

i.e. that S is weakly 2-free.

A similar result on Pythagorean quadruples (a, b, c, d) that says that exactly one of a, b, c are odd and d is odd will give us the result for $\mathbb{Q}^3$.

Conversely: for $\mathbb{Q}^4$: we have that

$$3\left(\frac{1}{6}, \frac{1}{6}, \frac{1}{2}, \frac{5}{6}\right) - 1\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) - 1(0,0,1,0) - 2(0,0,0,1) = (0,0,0,0),$$

while $3 - 1 - 1 - 2 = -1 \neq 0 \mod 2$. So the unit sphere here is not weakly 2-free, and thus $\mathbb{Q}^4$ is not 2-colorable.