The Unit Distance Graph	Instructor: Paddy
Lecture 3: $\chi(\mathbb{R}^2)$ and the Axiom of Choice	
Week 1 of 1	*Mathcamp 2010*

1 ZFC and ZFS: Different Models of Set Theory

On yesterday's problem set, we defined the following:

Axiom 1 (Axiom of Choice) *For every family Φ of nonempty sets, there is a choice function*

$$f : \Phi \to \bigcup_{S \in \Phi} S,$$

such that $f(S) \in S$ for every $S \in \Phi$.

So: back when this was first proposed as an axiom in 1910, many mathematicians fought it, on two grounds:

- Constructivist and intutionist mathematicians opposed it, on the grounds that it posits the existence of functions without any clue whatsoever as to how to find them!
- Many other working mathematicians just thought it was a true statement; i.e. that AC was a trivial consequence of any logical framework of mathematics.

Surprisingly enough, however, Paul Cohen and Kurt Gödel proved that the axiom of choice is independent of the Zermelo-Fraenkel axioms of set theory, the current framework within which we do mathematics: i.e. that it is its own proper axiom! Pretty much all of modern mathematics accepts the Axiom of Choice; it's a pretty phenomenally useful axiom, and most fields of mathematics like to be able to call on it when pursuing nonconstructive proofs.

There are, however, a number of disconcerting "paradoxes" that arise from working within ZFC, the framework of axioms given by the Zermelo-Fraenkel axioms + the axiom of choice:

- The well-ordering principle: the statement that any set S admits a well-ordering[1] Consequently, there's a way to order the real numbers so that they "locally" look like the natural numbers! Strange.

[1] A well-ordering on a set S is a relation $\leq$ such that the following properties hold:

- (antireflexive:) $a \leq b$ and $b \leq a$ implies that $a = b$.
- (total:) $a \leq b$ or $b \leq a$, for any $a, b \in S$.
- (transitive:) $a \leq b, b \leq c$ implies that $a \leq c$.
- (least-element:) Every nonempty subset of S has a least element.

- The Banach-Tarski paradox: there's a way to chop up and rearrange a sphere into two spheres of the same surface area.
- The existence of nonmeasurable sets: There are bounded subsets of the real line to which we cannot assign any notion of "length," given that we want length to be a translation-invariant, nontrivial, and additive function on $\mathbb{R}$.

Motivated by these strange results, Solovay (a set theorist) introduced the following two axioms:

- ($AC_{\aleph_0}$, the countable axiom of choice): For every **countable** family Φ of nonempty sets, there is a choice function

$$f : \Phi \to \bigcup_{S \in \Phi} S,$$

  such that $f(S) \in S$ for every $S \in \Phi$.

- (LM, Lebesgue-measurability): Every bounded set in $\mathbb{R}$ is measurable.

Theorem 2 (Solovay's Theorem) *There are models of mathematics in which ZF + LM + $AC_{\aleph_0}$ all hold.*

For brevity's sake, we will denote ZF + the axiom of choice by ZFC, and ZF + LM + $AC_{\aleph_0}$ by ZFS.

2 $\chi(\mathbb{R}^2)$ in ZFS

This discussion provokes a fairly natural question for this class: does $\chi(\mathbb{R}^2)$ depend on the axiom of choice? In other words, is $\chi^{ZFC}(\mathbb{R}^2)$ different from $\chi^{ZFS}(\mathbb{R}^2)$?

Well: as we currently don't know what $\chi^{ZFC}(\mathbb{R}^2)$ even *is,* answering this question completely seems to be a bit beyond our reach. However, the following two examples suggest that their chromatic numbers may be quite distinct:

Theorem 3 *Let G be the graph defined as follows:*

- $V(G) = \mathbb{R}$,
- $E(G) = \{(s,t) : s - t - \sqrt{2} \in \mathbb{Q}\}$.

Then $\chi^{ZFC}(G) = 2$.

Proof. Let

$$S = \{q + n\sqrt{2} | q \in \mathbb{Q}, n \in \mathbb{Z}\}.$$

Define an equivalence relation $\sim$ on $\mathbb{R}$ as follows: $x \sim y$ iff $x - y \in S$. Let $\{E_i\}_{i \in I}$ be the collection of all of the equivalence classes of $\mathbb{R}$ under $\sim$. Using the axiom of choice, pick one element y_i from each set E_i, and collect all of these elements in a single set E. Define the function $f : \mathbb{R} \to \mathbb{R}$ as follows:

$$f(x) = \text{ the unique element } y_i \text{ in } E \text{ such that } x \sim y_i.$$

Now define a two-coloring of $\mathbb{R}$ as follows: for any $x \in \mathbb{R}$, color x 1 iff there is an odd integer n such that

$$x - f(x) - n\sqrt{2} \in \mathbb{Q};$$

similarly, color x 2 iff there is an even integer n such that

$$x - f(x) - n\sqrt{2} \in \mathbb{Q}.$$

By construction, we know that $x \sim f(x)$; so $x - f(x)$ is always of the form $q + n\sqrt{2}$, and thus we always have exactly one of the two possibilities above holding. As well, if we examine any edge $\{x, y\}$, we have to have $x - y = q + \sqrt{2}$, for some q; i.e. $x \sim y$! So $f(x) = f(y)$, and thus we have that

$$\begin{aligned} x - y &= q + \sqrt{2} \\ \Rightarrow (x - f(x)) + (y - f(y)) &= q + \sqrt{2}; \end{aligned}$$

consequently, if both $x - f(x) - n\sqrt{2}$ and $y - f(y) - m\sqrt{2} \in \mathbb{Q}$, we must have one of n, m be odd and the other be even.

Theorem 4 *For G as above, $\chi^{ZFS}(G) > \aleph_0$.*

Proof. Consider the following lemma:

Lemma 5 *If $A \subset [0, 1]$ and A doesn't contain a pair of adjacent vertices in G, then A has measure[2] 0.*

Proof. So: consider the following rather large hammer from analysis, which we will use without proof:

Theorem 6 *(Lebesgue Density Theorem) If a set A has nonzero measure, then there is an interval I such that*

$$\frac{\mu(A \cap I)}{\mu(I)} \geq 1 - \epsilon,$$

for any $\epsilon > 0$.

[2] The measure of a set S is defined as the infimum of the sum $\sum(b_i, a_i)$, where we range over all collections of intervals $\{(a_i, b_i)\}$ such that $\bigcup(a_i, b_i) \supset S$. We denote this number by writing $\mu(S)$

So: choose any set A of measure > 0, and pick I such that

$$\frac{\mu(A \cap I)}{\mu(I)} \geq 99/100,$$

for instance. Then, pick $q \in \mathbb{Q}$ such that $\sqrt{2} < q < \sqrt{2} + \mu(I)/100$, and define $B = \{x - q + \sqrt{2} : x \in A\}$. Then B has been translated by at most $1/100$-th of the length of I: so we have that

$$\frac{\mu(B \cap I)}{\mu(I)} \geq 98/100.$$

So, because $(A \cap I) \cup (B \cap I) \subset I$, and both of these sets are almost all of I, we know that they must overlap! In other words, there's an element y in both A and B – but this means that there's an element y in A such that $y = x - q + \sqrt{2}$, with x *also* in A! i.e. there's a pair of elements x, y in A with an edge between them!

So: with this, our proof is pretty straightforward. Suppose that we could color $\mathbb{R}$ with $\aleph_0$-many colors, and that the collection of colors used is given by the collection $\{A_i\}_{i=1}^{\infty}$. Let $B_i = A_i \cap [0, 1]$; then we have that all of the B_i are disjoint and $\bigcup B_i = [0, 1]$. Consequently, we have that $\sum \mu(B_i) = \mu([0, 1]) = 1$; so at least one of the B_i's have to have nonzero measure! This contradicts our above lemma; consequently, no such $\aleph_0$-coloring can exist.