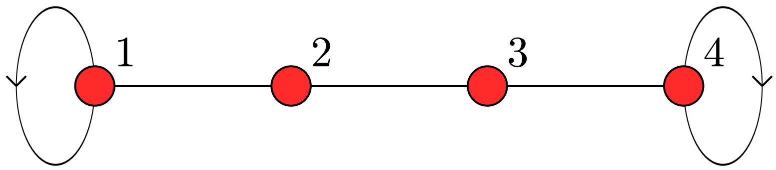
| Electrical Networks and Piffles | Instructor: Padraic Bartlett |
|---|---|
| Lecture 1: Electrifying Graphs and Piffles | |
| *Week 5* | *Mathcamp 2011* |

Oh nyo! While out on a walk, you accidentally left your piffle at vertex 2 in the following graph:




Piffles, left to their own devices, wander randomly between vertices, choosing between its neighbors randomly every minute and then walking to the chosen vertex. If our piffle makes it to vertex 4 (home,) it goes inside and is happy and safe! Conversely, if it makes it to vertex 1 (a black hole,) it gets sucked into an event horizon and is never seen to the outside universe again.

What are the piffle's chances of making it home? How can we model these kinds of behaviors?

# 1 Random Walks!

For a model as simple as this one, it's remarkably simple to determine what happens! Specifically, let's consider the piffle's chances of survival starting from **any** vertex $v$, not just 2: for notational convenience, denote this probability as $p(v)$. What do we know about these values?

- $p(4) = 1$: if the piffle starts at home, it's happy and safe!
- $p(1) = 0$: if we've accidentally left the piffle inside of the black hole, we're not going to see it anytime soon.
- For $v \neq 1, 4$, we have $p(v) = \frac{1}{2}p(v-1) + \frac{1}{2}p(v+1)$. If a piffle is at any vertex that's neither home or the black hole, it will choose between the two edges available to it with the same probability (1/2), and then go to that respective vertex via that edge. So, it's chances of survival are $\frac{1}{2}\cdot$ its chances at the vertex to its left, plus $\frac{1}{2}\cdot$ its chances at the vertex to its right.

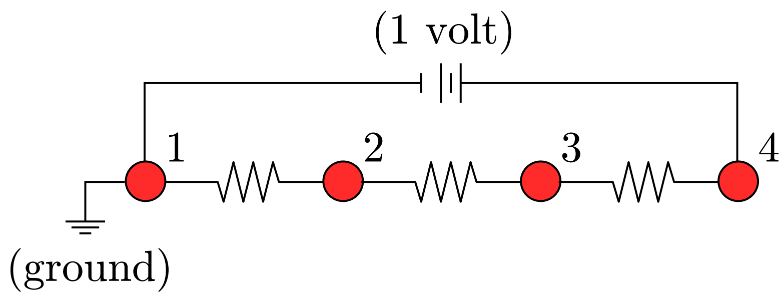
These are 4 equations

$$p(1) = 0, p(2) = \frac{p(1)+p(3)}{2}, p(3) = \frac{p(2)+p(4)}{2}, p(4) = 1$$

in four unknowns; solving these systems tells you that $p(2) = \frac{1}{3}, p(3) = \frac{2}{3}$, and thus that our specific piffle at vertex 2 only has a 1/3 chance of making it home. Poor piffle!

## 2 Electrifying Our Graphs

We drew such a pretty picture for the above example. You know what we should do? Electrocute it!




Specifically: take the same graph $P_4$ we were working with earlier, and do the following things to it:

1. Replace all of $P_4$'s edges with resistors of unit resistance 1.
2. Remove the loops at vertices 1 and 4.
3. Ground the vertex 1, and create a potential difference of 1 across the vertices 1 and 4.

If you've never ran into the concepts of voltage, current, conductance, or resistance before, that's cool! Think of voltage as a function from $V(G) \to \mathbb{R}$, current as a function $(V(G))^2 \to \mathbb{R}$, and resistance as a function $E(G) \to \mathbb{R}$ such that the following two properties are preserved:

- (Ohm's law:) The current across an edge $\{x, y\}$ in the direction $(x, y)$, $i_{xy}$, satisfies

$$i_{xy} = \frac{v(x) - v(y)}{R_{xy}},$$

  where $v(x), v(y)$ are the voltages at $x, y$ and $R_{xy}$ is the resistance of the edge $\{x, y\}$.

- (Kirchoff's law:) The sum of the currents into and out of any vertex other than the grounded vertex or the "source" vertex is zero: i.e. for any vertex neither grounded nor hooked up to power, we have

$$\sum_{y \in N(x)} i_{xy} = 0.$$

Furthermore, if we ground any vertex, its voltage is 0; if we have hooked up a vertex to k units of potential difference, its voltage is $k$. Also, for convenience's sake, we define the **conductance** of an edge $\{x, y\}$ as the reciprocal of its resistance: i.e. $C_{xy} = 1/R_{xy}$, and define the conductance of a vertex $x$ as the sum of the conductances of the edges leaving it: i.e. $C_x = \sum_{y \in N(x)} C_{xy}$.

In this setup, what happens? Well: we have that $v(1) = 0$, $v(4) = 1$, and for every vertex not 1 or 4,

$$\sum_{y \in N(x)} i_{xy} = 0;$$

i.e. for vertex 2, we have

$$\begin{aligned} 0 = \sum_{y \in N(x)} i_{2,y} = i_{2,1} + i_{2,3} &= \frac{v(2) - v(1)}{R_{xy}} + \frac{v(2) - v(3)}{R_{xy}} \\ &= v(2) - v(1) + v(2) - v(3), \end{aligned}$$

which implies that $v(2) = \frac{v(1)+v(3)}{2}$; similarly, we can derive that $v(3) = \frac{v(2)+v(4)}{2}$. In other words, to find the voltages at the vertices $2, 3$ we're solving the same equations we did for our piffle earlier: i.e. $v(2)$ is $1/3$, the probability that a piffle walking on our graph starting from 2 will make it to vertex 4 before vertex 1!

## 3 Electrons Are Piffles

Surprisingly, this property above – that our random walk and electrical network were, in some sense, the "same" – holds for all graphs! In the following lemmas, we make this idea concrete:

**Lemma 1** *Suppose that we have a graph $G$. Define a **piffle** starting at a vertex $x$ in our graph as the following process:*

- *Initially, the piffle starts at $x$.*
- *Every minute, if a a piffle is at some vertex $w$, it randomly chooses one of the elements $y \in N(w)$ with equal probability – i.e. each neighbor has probability $1/\deg(w)$ of being picked – and goes to that vertex.*

*Let $a, b$ be a pair of distinguished vertices in our graph, and $p(x)$ be the probability that a piffle starting at the vertex $x$ will make it to vertex $b$ before vertex $a$.*

*Then $p(x) = v(x)$, if we turn our graph $G$ into a electrical network with $a$ connected to ground, a unit of electrical potential sent across $a$ and $b$, and replace every edge with $G$ with a resistor with resistance 1.*

**Proof.** This is pretty much identical to what we just did. Specifically: we know from Ohm's law that

$$i_{xy} = \frac{v(x) - v(y)}{R_{xy}};$$

therefore, if we plug Ohm's law into Kirchoff's law, we have that whenever $x \neq a, b$, we have

$$\begin{aligned}
\sum_{y \in N(x)} \frac{v(x) - v(y)}{R_{xy}} &= v(x) \cdot \left( \sum_{y \in N(x)} \frac{1}{R_{xy}} \right) - \sum_{y \in N(x)} \frac{v(y)}{R_{xy}} \\
\Rightarrow \quad & v(x) \cdot \left( \sum_{y \in N(x)} \frac{1}{R_{xy}} \right) = \sum_{y \in N(x)} \frac{v(y)}{R_{xy}} \\
\Rightarrow \quad & v(x) C_x = \sum_{y \in N(x)} C_{xy} v(y) \\
\Rightarrow \quad & v(x) = \sum_{y \in N(x)} \frac{C_{xy}}{C_x} v(y).
\end{aligned}$$

But what is $\frac{C_{xy}}{C_x}$? It's the probability that we choose the neighbor $y$ out of $N(x)$, if we're picking neighbors of $x$ with probabilities given by $1/R_{xy}$. In this specific case, where all of our resistances are 1, this is just the chance that a piffle at vertex $x$ will go to $y$ in our random walk!

But this is the exact same equation we're asking $p(x)$ to satisfy: i.e. we want

$$p(x) = \sum_{y \in N(x)} (\text{chance piffle goes from x to y}) \cdot p(y).$$

The only other restrictions we have on our voltage or random walk is that $v(a) = p(a) = 0, v(b) = p(b) = 1$: in other words, the equations that we're asking our voltage function to satisfy are the same that we're asking our probability function to satisfy!

As there is only one sensical probability distribution or voltage distribution for any network[1], these must be the same function!

We have an excellent interpretation of voltage in terms of our random walk. For fun, let's find another:

**Lemma 2** *Take the same setup as before, and let $u(x)$ denote the average number of times that a piffle starting at vertex $b$ will wander through the vertex $x$ before reaching $a$. Then, if we have a potential difference of $u(b)/C_b$ from $a$ to $b$ where $a$ is again grounded, we have*

$$v(x) = \frac{u(x)}{C_x}.$$

**Proof.** Trivially, we have $v(a) = 0 = u(a)$ and $v(b) = u(b)/C_b$. Also, for every vertex $x \neq a, b$, we have

$$u(x) = \sum_{y \in N(x)} (\text{chance of going from y to x}) \cdot u(y);$$

[1] If you are unpersuaded here, prove this! In other words: prove that if I have two equations that satisfy the relations we've demonstrated here, that they *must* be equal. The concept behind this equality is the idea of "harmonic" functions, something that is both beautiful and beyond the scope of this class.

i.e. the number of times we'll go through $x$ is clearly the sum of the number of times we'll go through the neighbors of $x$, weighted by the chances of going from $y$ to $x$.

From our above discussion, we know that the chance of going from $y$ to $x$ is $\frac{C_{xy}}{C_x}$: i.e.

$$u(x) = \sum_{y \in N(x)} \frac{C_{xy}}{C_x} \cdot u(y).$$

Let $P_{yx}$ denote this chance of going from $y$ to $x$. We know that because $C_{xy} = C_{yx}$, $C_x P_{xy} = C_y P_{yx}$: plugging this into the above equation, we have

$$u(x) = \sum_{y \in N(x)} \frac{P_{xy} C_x}{C_y} \cdot u(y)$$

$$\sum_{y \in N(x)} \frac{P_{xy} C_x}{C_y} \cdot u(y)$$

$$\Rightarrow \qquad \frac{u(x)}{C_x} = \sum_{y \in N(x)} P_{xy} \cdot \frac{u(y)}{C_y}.$$

But these are (once again) the same equations as voltage! So we have $v(x) = u(x)/C_x$.

We mention this specifically to motivate the idea of **current**. Mentally, how do we think of current? In a super-simplistic sense, we can model the current flowing through an oriented edge $(x, y)$ as the "flow" of electrons from $x$ to $y$: i.e. if we have electrons randomly bumbling about on our graph starting at $b$ and wandering around until they get to $a$, we might hope that $i_{xy}$ is the average number of electrons that go from $x$ to $y$, minus the number that go "backwards" along this edge from $y$ to $x$.

We prove this here:

**Lemma 3** *Let $G$ be a graph as above, with potential difference still $u(b)/C_b$. Then we have that*

$$i_{xy} = u(x)P_{xy} - u(y)P_{yx};$$

*i.e. the current from $x$ to $y$ is equal to the average number of paths to $x$ times the probability of going from $x$ to $y$, minus the average number of paths to $y$ times the probability of going from $y$ to $x$; i.e. average number of walks that go from $x$ to $y$, minus the number that go "backwards" along this edge from $y$ to $x$.*

**Proof.** We simply calculate, using our identities:

$$i_{xy} = (v(x) - v(y)) \cdot C_{xy} = \left( \frac{u(x)}{C_x} - \frac{u(y)}{C_y} \right) C_{xy} = u(x) \cdot P_{xy} = u(y)P_{yx}.$$

Electricity! Random walks! Apparently, mostly the same. Tomorrow, we'll use this to answer in a stupidly elegant way the following problem posed by Polya:

**Question 4** *Suppose you've lost your piffle again, but this time you've lost him at a random point in the integer lattice $\mathbb{Z}^d$! Suppose your home is at the origin. Given enough time, will the piffle return home? Or is there a nonzero chance that it will wander the desolate wastelands of $\mathbb{Z}^d$ forever?*