Electrical Networks and Piffles | Instructor: Padraic Bartlett

Lecture 2: Piffles in $\mathbb{Z}^d$

Week 5 | *Mathcamp 2011*

Yesterday, we showed that electrical networks and random walks are remarkably related subjects. Today, I want to use these connections to study a question of Polya:

Question 1 *Suppose you've lost a piffle (i.e. a random walker) at some point in the integer lattice $\mathbb{Z}^d$! Suppose your home is at the origin. Given enough time, will the piffle return home?*

1 Circuits as Black Boxes

To attack this kind of question, it might help to introduce some new ideas. Specifically, suppose that we have a circuit with two points a, b, where we've grounded a and have a voltage v established at b. Then there is some amount of current flowing out of b: this current i_b is the sum of the currents $\sum_{x \in N(b)} i_{bx}$.

Mentally, we can think of this entire circuit as just a large and bulky resistor – we have applied a voltage across two points, and a current is flowing across the circuit. Specifically, if we think of this object as a large resistor, we know that its resistance can be found by applying Ohm's law: call this quantity R_{eff}, and set it equal to $v(b)/i_b$. Similarly, define $C_{\text{eff}} = 1/R_{\text{eff}}$.

Earlier, we noted that the current across an edge (x, y) was proportional to the expected number of paths from x to y minus the expected number of paths from y to x, if the voltage we put out of b was equal to 1 (say.) Does this idea still hold here? Well: calculating, we have

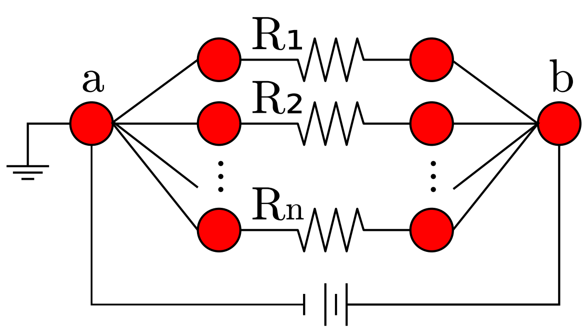
$$\begin{aligned}
i_b &= \sum_{x \in N(b)} (v(b) - v(y)) \cdot C_{by} \\
&= \sum_{x \in N(b)} (v(b) - v(y)) \cdot \frac{C_{by}}{C_b} \cdot C_b \\
&= C_b \left(v(b) \sum_{x \in N(b)} \frac{C_{by}}{C_b} - \sum_{x \in N(b)} \frac{C_{by} v(y)}{C_b} \right) \\
&= C_b \left(1 - \sum_{x \in N(b)} v(y) \frac{C_{by}}{C_b} \right) \\
&= C_b \left(1 - \sum_{x \in N(b)} v(y) P_{by} \right).
\end{aligned}$$

What is this last quantity? Well: P_{by} denotes the probability of going from b to y, and (as we saw yesterday) $v(y)$ is the probability that a walk starting at y will make it to b before a. So: if we're starting at b and leaving to any of b's neighbors (which we pick with probability P_{by}), the chances of returning to b before making it to a is just $v(y)$. Therefore, the sum on the inside of our parentheses is precisely the chances of starting at b and returning there before making it to a: therefore, the entire quantity is just C_b multiplied by the chances of a walk starting at b and making it to a before returning to b. Call the chance of this occuring p_{esc}: then, we have just shown that

$$\frac{i_b}{C_b} = p_{\text{esc}}.$$

2 Resistance: Surprisingly Not Futile

This, basically, is **win**. Specifically: we know how to find resistances! Super-specifically: suppose we have a series of resistors connected "in parallel," i.e. like in the picture below:

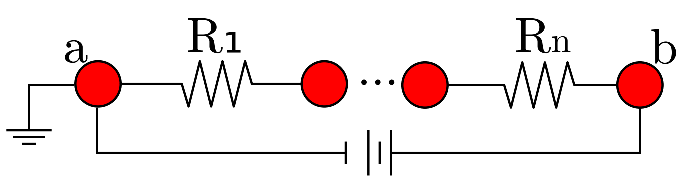


Then the effective resistance of the pictured circuit is the reciprocal of the sum of the reciprocals of the resistors:

$$\frac{1}{R_{\text{eff}}} = \sum_{i=1}^{n} \frac{1}{R_i}.$$

Alternately, you can think of this claim as the statement that the "effective conductance" of the circuit is the sum of the conductances of the circuit.

Similarly, suppose we have a circuit made of resistors linked in series, as depicted below:



Then the effective resistance of the pictured circuit is the sum of the resistors:

$$R_{\text{eff}} = \sum_{i=1}^{n} R_i.$$

It bears noting that you can deduce these properties from the two rules we've stated for electrical networks, Ohm's law and Kirchoff's law; the first property just says that the conductances sum when we have resistors in parallel, and the second says that resistances sum when we have resistors in series. We omit a formal proof here, but it's not remotely difficult.

The other property of electrical networks we're going to use throughout our proofs is Rayleigh's Monotonicity Theorem, which we state here:

Theorem 2 *If any of the individual resistances in a circuit increase, then the overall effective resistance of the circuit can only increase or stay constant; conversely, if any of the individual resistances in a circuit decrease, the overall effective resistance of the circuit can only decrease or stay constant.*

In specific, cutting wires (setting certain resistances to infinity) only increases the effective resistance, while fusing vertices together (setting certain resistances to 0) only decreases the effective resistance.

We also omit the proof of the statement here; it's actually somewhat involved, but can be proven directly from our two laws without any appeal to electrical networks in "reality."

3 Piffles in $\mathbb{Z}^d$

Given these tools, we are now equipped to tackle our question! Let's turn to $\mathbb{Z}^1$, as a quick warm-up. Our question, then, is whether a piffle starting at some point on the lattice (say the origin) will always return to the origin, or whether there's a nonzero chance that it wanders off forever.

We only have the tools to talk about finite graphs. To turn these tools into ones to deal with an infinite connected graph G, do the following:

- Let x be whichever node we're designating as the origin, and $G^{(r)}$ be the graph formed by taking all of the vertices connected to x by paths of length at most r.
- Turn this into a electrical network problem by soldering all of the vertices that are distance r from x together into one big ball, grounding them, and putting one unit of voltage at x, and making all of the edges resistors with resistance 1. Then, via our earlier discussions, we can talk about the probability that a piffle starting at x will make it to distance r before returning to x: denote this quantity as $p_{\text{esc}}^{(r)}$.
- Let p_{esc} be the limit $\lim_{r\to\infty} p_{\text{esc}}^{(r)}$. If this is nonzero, then there is some nonzero chance that our piffle will wander forever; if this is zero, then our piffle must eventually return to the origin.
- Notice that if it must eventually return to the origin, then it must eventually make it to any vertex w in G! This is because starting from the origin, we always have some nonzero chance to make it to w, and (because we return to the origin infinitely many times) we get infinitely many tries.

If G is a graph on which we return infinitely many times to the origin, we call G **recurrent**; if it is a graph where there is a chance that we will never return to the origin, we call G **transient**.

Theorem 3 *The one-dimensional lattice graph $\mathbb{Z}$ is recurrent.*

Proof. Let 0 be the origin, without any loss of generality. Using our earlier discussion, we know that

$$p_{\text{esc}}^{(r)} = \frac{i_0}{C_0} = \frac{1}{C_0} \cdot \frac{v(0)}{R_{\text{eff}}} = \frac{1}{C_0 R_{\text{eff}}}.$$

We know that the resistance of a string of r resistors in a row is r, from our earlier discussion about resistors in series: as there are two such strings in parallel, we know that their combined resistance is $\frac{r^2}{2r} = \frac{r}{2}$, and therefore that (because the capacitance of the origin is 2)

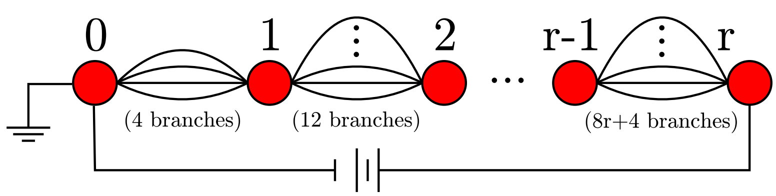
$$p_{\text{esc}}^{(r)} = \frac{1}{r}.$$

The limit as r goes to infinity of this quantity is 0; therefore, this walk is recurrent.

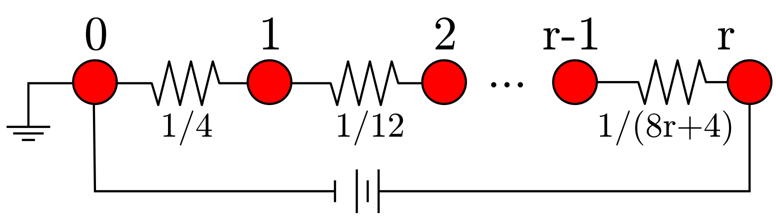
Theorem 4 *The two-dimensional lattice graph $\mathbb{Z}^2$ is recurrent.*

Proof. Take our graph, turn it into an electrical network with origin $= (0, 0)$, and perform the following really clever trick: for every r, let V_r be the collection of all of the vertices that are distance r from the origin under the taxicab metric (i.e. shortest length of a path.) Take our graph and **short** all of V_r's vertices into one huge clump, for each r: i.e. take the collection of all of the vertices at distance r, and just stick them all together! We know that this reduces the overall resistance, because of Rayleigh's principle; therefore, we know that if this graph is recurrent, $\mathbb{Z}^2$ must be as well.

What does this processs do to the restricted graph $(\mathbb{Z}^2)^{(r)}$? Well, it produces the following picture:



What is the resistance here? Well: there are $8n+4$ resistors between node n and node $n+1$; therefore, this graph is equivalent to the path on $\{0, \ldots r\}$ where the resistance between n and $n+1$ is $\frac{1}{8n+4}$:



Therefore, we can see that the limit of the resistance of these r-restricted graphs is the sum

$$\sum_{i=1}^{r} \frac{1}{8i+4},$$

which goes to infinity; therefore, the current on these graphs and thus the $p_{\text{esc}}^{(r)}$'s go to 0. So this graph is also recurrent.

Lemma 5 *Suppose C is a circuit with two vertices x, y that are not connected by a resistor and are at the same potential: i.e. $v(x) = v(y)$. Then shorting together x and y does not change the voltages or currents in the circuit.*

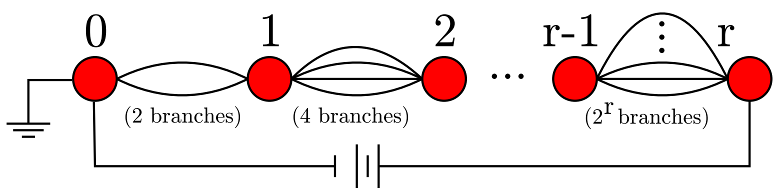
Proof. HW!

Theorem 6 *The three-dimensional lattice graph $\mathbb{Z}^3$ is transient.*

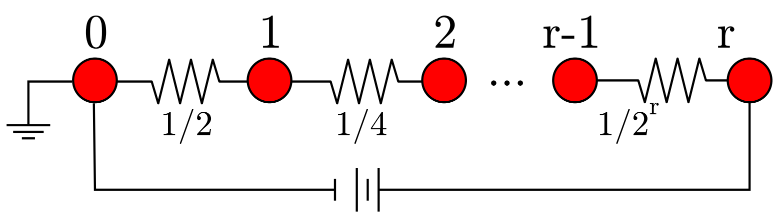
Proof. For $\mathbb{Z}^2$, the trick we used was to "short" a bunch of vertices together, and show that the resulting graph (which was simpler, even though its resistances were "lower") was recurrent. Here, in $\mathbb{Z}^3$, we're going to "cut" a number of resistors, and show that the resulting (simpler, higher-resistance) graph is transitive! (The normal proof of this theorem is much more difficult without these observations; it's only with this "shorting" and "cutting" that we can pull this off with such relative ease[1].

Specifically: lattices are **hard** to calculate resistances on. Let's try something simpler: a tree!

Well, we should be careful what tree we mean For example, consider the infinite binary tree graph T_2, where each edge is a resistor of resistance 2. Notice that (by symmetry) all of the nodes at any fixed distance k from the origin have the same potential: therefore, we can short them all together without changing anything in our graph. If we take this graph and cut it off at its first r nodes, we have the following picture:



We can therefore use our earlier observations on resistors in parallel to turn this into the following circuit:



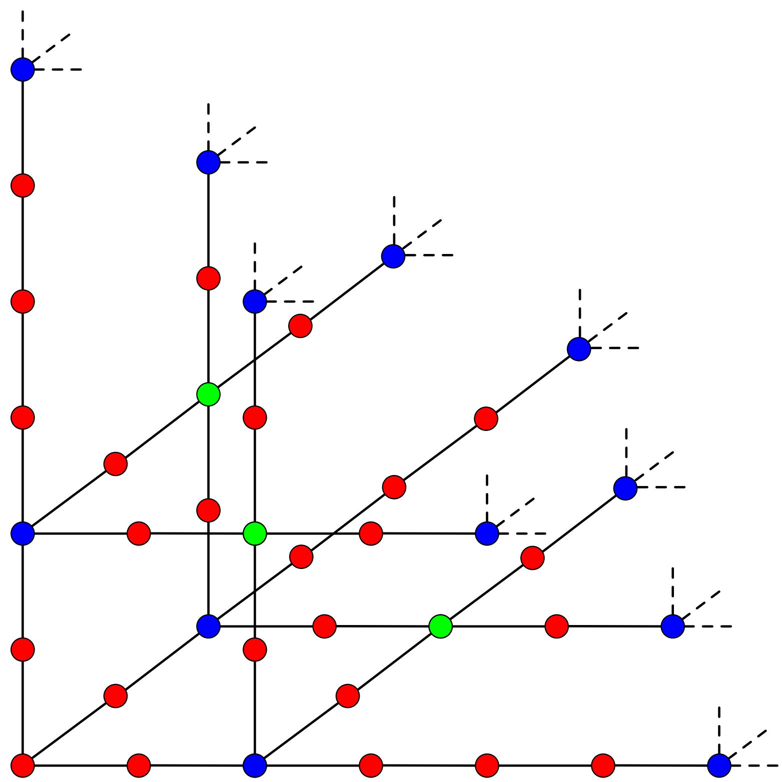
[1]Insert your own "short-cut" pun here.

This has resistance $\sum 1/2^n = 1$. Winning!

Kind-of. See, the number of nodes in a binary tree grows like 2^n, whereas the number of nodes distance n from the origin grows like n^2: so we're never going to be able to find a binary tree in $\mathbb{Z}^3$! Whatever will we do?

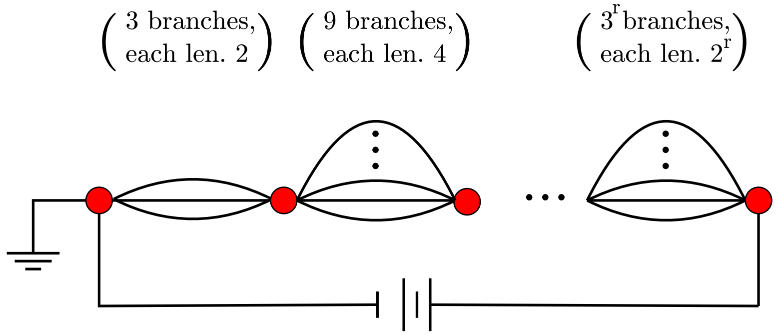
We will be **clever**. Specifically, let's stay with the tree structure. Binary was overkill: the sum $\sum 1/2^n$ converges far faster than we need! We just need a tree who splits often enough that we'll get *some* sort of convergent thing at the end of the day.

To do this, consider the following kind of"tree:"



As currently drawn: not a tree. However, if you pretend that each of the green nodes are "doubled", by creating two vertices at each of those locations and passing only one branch through each node, it's a tree! Furthermore, because these nodes are at the same distances from the origin, we know that they have the same voltage passing through them by symmetry! This tree branches at distances $\sum_{n=1}^{r} 2^n$ for every $r \geq 1$ (i.e. at the blue nodes,) and creates three branches (one in the positive x, y and z directions) at each such distance. By construction, these branches never intersect at these "branching" blue nodes: therefore, this tree is realizable in $\mathbb{Z}^3$ as depicted above.

By identifying nodes of distance $\sum_{n=1}^{r} 2^n$ for every n from the origin, the graph on this tree restricted to the distances $\sum_{n=1}^{r} 2^n$ is equivalent to a circuit of the form



By applying our known results about resistors in series and parallel, we can see that the total resistance between any two nodes $n-1, n$ in the above circuit is

$$\frac{2^n}{3^n};$$

therefore, our tree at stage R has total resistance

$$\sum_{n=1}^{r} \frac{2^n}{3^n} = \frac{1-(2/3)^{r+1}}{1-(2/3)} - 1.$$

As r goes to infinity, this goes to 2; therefore, the current $i_b = v(b)/R_{\text{eff}} = 1/2$ at infinity is positive, and consequently the value $p_{\text{esc}} = i_b/C_b = \frac{1/2}{3} = 1/6$ is positive and nonzero. Therefore, by our earlier discussion, there is a nonzero chance of escape! In other words, our random walker may never return to the origin

This allows us to finally answer the question we've designed this class around:

Corollary 7 *If you've lost a piffle somewhere on an integer lattice, it will come back home if and only if it cannot fly.*