| Flows in Graphs | Instructor: Padraic Bartlett |
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# Lecture 4: An Introduction to Algebraic Flows

*Week 2* — *Mathcamp 2011*

This lecture, unlike some of the earlier ones, will focus on covering a lot of the main theorems about algebraic flows: we'll prove some statements here, but for the most part we'll sketch the outlines of proofs instead and leave the details for the HW.

## 1 Definitions and Fundamental Results

From the earlier HW sets, recall the definitions of $A$-circulation and $A$-closure, where $A$ is an abelian group (most often, the integers modulo some value $k$:)

**Definition.** $(*)$ A directed graph $G$ has an **$A$-circulation** $g$ iff there is some way to assign elements of $A$ to every edge in $G$, such that we obey Kirchoff's law at every vertex $v$ in $G$:

$$\sum_{w \in N^+(v)} g(v,w) - \sum_{w \in N^-(v)} g(w,v) = 0.$$

We say that an $A$-circulation is in fact an **$A$-flow** iff $g(e) \neq 0$, for every edge $e \in E(G)$.

Notice that in the above definition, the direction of $G$ is not terribly important: if we have a different graph $G$ where some of the edges $e \in E(G)$ were reversed, this would just correspond to switching the sign of $g(e)$ on those edges. Therefore, we'll often just refer to $A$-circulations as maps on undirected graphs $G$, that have the additional property that $f(x,y) = -f(y,x)$; in this sense, we're offloading the process of assigning directions of edges to the flow itself (because the negative signs are effectively doing this already.)

We state a few surprising results here, the first of which was on your HW yesterday:

**Theorem 1** *For any graph $G$, there is a polynomial $P$ such that the number of distinct $A$-flows on $G$ is given by $P(n-1)$, where $|A| = n$.*

**Proof.** (Sketch:) Work by induction on $|E|$. Take any edge $e_0$ in your graph, and consider the two graphs $G \setminus \{e_0\}$, where you just delete the edge, and $G/\{e_0\}$, where you shrink the edge to a point. To each of these graphs, apply our inductive hypothesis, to get a pair of polynomials $P_2(x)$ and $P_1(x)$; finally, use a counting argument on the $A$-flows of $G$ to show that $P_2(n) - P_1(n)$ is the number of distinct $A$-flows on $G$.

The main reason we mention the above result is that it, in some sense, says that our choice of group for a $A$-flow doesn't really matter! – the only important thing is how many elements it has in it. To formally state this as a corollary:

**Corollary 2** *If $A$, $B$ are a pair of groups with $|A| = |B|$, and $G$ is a graph, then there is an $A$-flow on $G$ if and only if there is a $B$-flow on $G$.*

In a sense, we have reduced our study of $A$-flows to the study of $\mathbb{Z}/k\mathbb{Z}$-flows, which should make our life remarkably easier. One thing we might wonder, now, is whether we can turn these $\mathbb{Z}/k\mathbb{Z}$-flows into actual $\mathbb{Z}$-flows: i.e. whether by being clever, we can transform the weaker requirement of

$$\sum_{w\in N^+(v)} g(v,w) - \sum_{w\in N^-(v)} g(w,v) \equiv 0 \mod k$$

into the stronger requirement

$$\sum_{w\in N^+(v)} g(v,w) - \sum_{w\in N^-(v)} g(w,v) = 0.$$

As it turns out, we can!

**Definition.** A $k$-flow $f$ on a graph $G$ is a mapping $f : E(G) \to \mathbb{Z}$, such that $0 < |f(e)| < k$ and $f$ satisfies Kirchoff's laws at every vertex.

**Theorem 3** *A graph $G$ has a $k$-flow iff it has a $\mathbb{Z}/k\mathbb{Z}$-flow.*

**Proof.** (Sketch:) Any $k$-flow on $G$ satisfies the relation

$$\sum_{w\in N^+(v)} g(v,w) - \sum_{w\in N^-(v)} g(w,v) = 0.$$

at every vertex $v$. Therefore, this flow trivially induces a $\mathbb{Z}/k\mathbb{Z}$-flow by interpreting all of its values as elements of $\mathbb{Z}/k\mathbb{Z}$ (because if a sum of things is equal to 0, it's certainly equal to $0 \mod k$.)

To go the other way: take any flow $f : E(G) \to \mathbb{Z}/k\mathbb{Z}$. Interpret $f$ as a function $E(G) \to \{1, \ldots k-1\}$. Then all we have to do is for every edge $e$, decide whether we map it to $f(e)$ or $f(e) - k$, in a sufficiently consistent way that we insure Kirchoff's laws are still obeyed.

It turns out that if you pick an interpretation that minimizes the Kirchoff-sums $\left|\sum_{w\in N^+(v)} g(v,w) - \sum_{w\in N^-(v)} g(w,v)\right|$ at every vertex, it actually does this! (You can show this by contradiction; it's not terribly surprising, and is mostly sum-chasing.

## 2 Some Calculations

To recap: we've proven that for an abelian group $A$ with $|A| = k$, a graph $G$ has an $A$ flow iff it has a $\mathbb{Z}/k\mathbb{Z}$-flow iff it has a $k$-flow. Therefore, in a sense, the only interesting questions to be asked here is the following: given a graph $G$, for what values of $k$ does $G$ have a $k$-flow?

We classify some cases here:

**Proposition 4** *A graph $G$ has a 1-flow iff it has no edges.*

**Proof.** This is trivial, as a 1-flow consists of a mapping of $G$'s edges to 0, such that no edge is mapped to, um, 0.

**Proposition 5** *A graph $G$ has a 2-flow iff the degree of all of its vertices are even.*

**Proof.** A 2-flow consists of a mapping of $G$'s edges to $\{\pm 1\}$, or equivalently a mapping of $G$'s edges to 1 in $\mathbb{Z}/2\mathbb{Z}$, in a way that satisfies Kirchoff's law. However, we trivially satisfy Kirchoff's laws in the $\mathbb{Z}/2\mathbb{Z}$ case iff the degree of every vertex is even; so we know that this condition is equivalent to having a 2-flow.

**Definition.** For later reference, call any graph where the degrees of all of its vertices are even a **even** graph; relatedly, call any graph where the degree of any of its vertices is 3 a **cubic** graph.

**Proposition 6** *A cubic graph $G$ has a 3-flow iff it is bipartite.*

**Proof.** A 3-flow consists of a mapping of $G$'s edges to $\{1, 2\}$ in $\mathbb{Z}/3\mathbb{Z}$, in a way that satisfies Kirchoff's law. Suppose that we have some such flow on our graph: call it $f$.

Take any cycle $(v_1, v_2, , v_n)$ in this graph, and consider any two consecutive edges $(v_1, v_2), (v_2, v_3)$. Suppose $f$ assigned these the same value, and let $w$ be $v_2$'s third distinct neighbor: then, by Kirchoff's law, we know that

$$-f(v_1, v_2) + f(v_2, v_3) + f(v_2, w) = 0 \Rightarrow f(v_2, w) = 0.$$

But $f$ is a $\mathbb{Z}/3\mathbb{Z}$-flow: so this cannot happen! Therefore, the values 1 and 2 have to occur alternately on this cycle, and therefore it must have even length. Having all of your cycles be of even length is an equivalent condition to being bipartite, so we know that our graph must be bipartite.

To see the other direction: suppose that $G$ is bipartite, with bipartition $V_1, V_2$. Define $f(x, y) = 1$ and $f(y, x) = -1 = 2$, for all $x \in V_1, y \in V_2$; this evaluation means that for any $x \in V_1$, we have

$$f(x, y_1) + f(x, y_2) + f(x, y_3) = 1 + 1 + 1 \equiv 0 \mod 3$$

and for any $y \in V_2$, we have

$$f(y, x_1) + f(y, x_2) + f(y, x_3) = 2 + 2 + 2 \equiv 0 \mod 3$$

by summing over their three neighbors in the other part of the partition. So this is a 3-flow, and we've proven the other direction of our claim.

For simplicity's sake, we introduce the function $\varphi(G)$ to denote the smallest value of $k$ for which $G$ admits a $k$-flow: if no such value exists, we say that $\varphi(G) = \infty$.

We now list here a series of claims whose proofs we leave for the HW:

**Proposition 7** $\varphi(K_2) = \infty$.

**Proposition 8** *More generally, a connected graph $G$ has $\varphi(G) = \infty$ whenever $G$ has a **bridge**: i.e. there is an edge $e \in E(G)$ such that removing $e$ from $G$ disconnects $G$.*

**Proposition 9** $\varphi(K_4) = 4$.

**Proposition 10** *If $n$ is even and not equal to 2 or 4, then $\varphi(K_n) = 4$.*

**Proposition 11** *A graph $G$ has a 4-flow if and only if we can write it as the* ***union*** *of two even graphs: i.e. there are a pair of graphs $G_1, G_2$, with possibly overlapping edge and vertex sets, such that $V(G) = V(G_1) \cup V(G_2), E(G) = E(G_1) \cup E(G_2)$.*

**Proposition 12** *A cubic graph $G$ has a 4-flow if and only if it is three-edge colorable.*

**Corollary 13** *The Petersen graph has no 4-flow.*

On the HW, you (hopefully) showed that the Petersen graph does have a 5-flow earlier in the week; therefore, we know that $\varphi(\text{Pete}) = 5$.

# 3 Open Conjectures

Surprisingly, when you start computing more of these numbers, it seems pretty much impossible to find anything that's got a value of $\varphi$ bigger than 5 and not yet infinity. This motivated Tutte to make the following conjectures on $k$-flows, which are still open to this day!

**Conjecture 14** *Every bridgeless graph has a 5-flow.*

More surprisingly, it seems like the Petersen graph, in a sense, is unique amongst graphs that have 5-flows:

**Conjecture 15** *Every bridgeless graph that doesn't contain the Petersen graph as a minor has a 4-flow.*

Currently, the best known result is a theorem of Seymour, from 1981, that proves that every bridgeless multigraph has a 6-flow: due to the lack of time, we omit this proof here. (Interested students should come and talk to me if they want to see it!) Still, the gap between 6 and 5 is wide-open, as is the improvement to 4 in the case of graphs that don't contain the Petersen graph as a minor.