Flows in Graphs | Instructor: Padraic Bartlett

Homework 2

Week 2 | *Mathcamp 2011*

Attempt the problems that seem interesting! Easier exercises are marked with $(-)$ signs; harder ones are marked by $(*)$. Open questions are denoted by writing $(**)$, as they are presumably quite hard. Oh! Also, typos build character: if you find any (not that there ever could be such things in my problem sets,) correct them to the most reasonable thing you can think of and proceed from there!

1. Yesterday, we didn't get quite far enough in lecture to attack many of the problems on the HW; so if there are problems you still have left there, return to those!

2. Consider the following problem:

   - You're going on a hardcore hike, and you're trying to decide what items to take with you!
   - However, some of these items are only useful if they're brought along with others. For example, a can of soup is only useful if you bring along a bowl, a spoon, and some means of heating the soup; your left shoe is kind of useless without a matching right shoe, and so on/so forth.
   - As well, each of these items has an associated **cost**, in terms of their weight; for example, you probably don't want to bring a giant cooler jug of water with you, as its cost probably overwhelms its benefit. How do you decide what items to bring with you?
   - To formalize this mathematically: suppose you have a set $J = \{j_1, \ldots j_n\}$ of items, each item $j_i \in J$ has a cost c_i, a collection of "useful" sets of items $S = \{S_1, \ldots S_k : S_i \subseteq J\}$, and an associated benefit b_i to each set S_i. What collection of items $K \subset J$ will maximize our cost-benefit ratio (i.e. $\sum_{S_i \subset K} b_i - \sum_{j_i \in K} c_i$?)

   Specifically: given any such sets J, S and cost/benefit pairings, create a network G on which a minimal cut corresponds to a optimal choice of items.

3. Create some sample item/cost lists and run the algorithm above to actually find some optimal choices!

4. Using Max-Flow Min-Cut, prove the vertex form of Menger's theorem: if s, t are distinct nonadjacent vertices in a graph G, then the minimal number of vertices needed in a set S such that any path from s to t travels through an element in S is equal to the maximal number of vertex-disjoint paths from s to t.

5. Prove the König-Egevary theorem via Max-Flow Min-Cut: Let G be a bipartite graph. Let the size of the largest set of disjoint edges in G – in other words, the size of the largest matching in G – be denoted by $\alpha'(G)$. Let the size of the smallest collection of vertices such that every edge is incident to at least one vertex – i.e. the size of the smallest vertex cover of G – be denoted by $\beta(G)$. Then $\alpha'(G) = \beta(G)$.