Flows in Graphs	Instructor: Padraic Bartlett

Homework 3

Week 2 — *Mathcamp 2011*

Attempt the problems that seem interesting! Easier exercises are marked with $(-)$ signs; harder ones are marked by $(*)$. Open questions are denoted by writing $(**)$, as they are presumably quite hard. Oh! Also, typos build character: if you find any (not that there ever could be such things in my problem sets,) correct them to the most reasonable thing you can think of and proceed from there!

1. (a) $(-)$ Prove the following extension of the Max-Flow Min-Cut Theorem:

   Theorem 1 *Suppose that G is a directed graph with source and sink nodes s, t. Suppose further that G comes with a pair of rational capacity functions $l, u : E(G) \to \mathbb{Q}^+ \cup \{\infty\}$ such that $l(e) \leq u(e)$, and a* ***feasible*** *flow f_0 (i.e a flow such that $l(e) \leq f(e) \leq u(e)$.) Then there is a feasible flow f on G and cut S on G such that*

   - *for any $x \in S, y \notin S$, we have $f(x,y) = u(x,y)$, and*
   - *for any $x \notin S, y \in S$, we have $f(x,y) = l(x,y)$.*

   (b) $(-)$ Prove that the above extension still holds if we allow our capacity functions and flows to attain negative values.

   (c) $(-)$ Using the above two results, show that given any lower bound on capacities $l(e)$, we can find a **minimal flow** and **maximal cut** with respect to this lower bound l: i.e. prove a Min-Flow Max-Cut theorem.

2. Using observation 1(a), consider the following problem: suppose we have a matrix of real numbers, like

   $$\begin{pmatrix} 2.5 & 3.3 & 1.4 \\ 3.1 & .1 & 0.5 \\ 2.1 & 3.3 & 0.4 \end{pmatrix}$$

   Something we will often want to do with these kinds of things is **round** them: i.e. to round each of the entries in this matrix to either its floor or ceiling (making possibly different choices for each one.) However, we'd like to do this in a way that preserves the **row** and **column** sums of our matrix: i.e. in such a way that the resulting integer matrix's row and column sums agree with the original matrix's sums, up to rounding.

   As it turns out, you can solve this with flows! Create a network with lower and upper capacity functions l, u such that any feasible flow on this network is a solution to the above problem.

3. $(*)$ When do you have such a feasible flow? Calculate with a few examples.