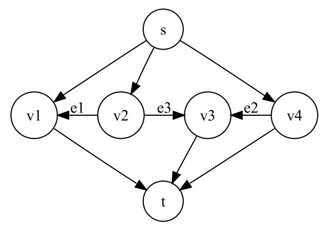
Flows in Graphs — Instructor: Padraic Bartlett

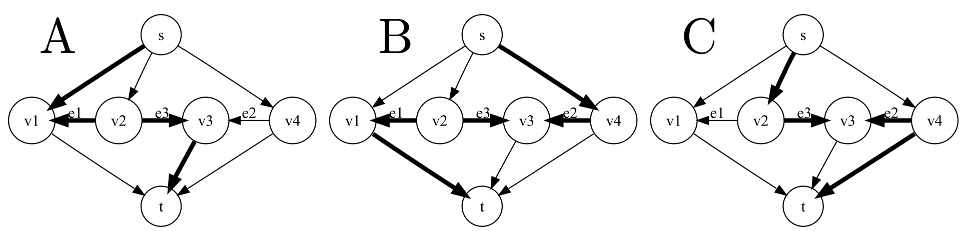
Homework 4

Week 2 — *Mathcamp 2011*

Attempt the problems that seem interesting! Easier exercises are marked with $(-)$ signs; harder ones are marked by $(*)$. Open questions are denoted by writing $(**)$, as they are presumably quite hard. Oh! Also, typos build character: if you find any (not that there ever could be such things in my problem sets,) correct them to the most reasonable thing you can think of and proceed from there!

1. Consider the graph below, which we've drawn four distinct paths on:





Suppose that the capacity function on this graph has $c(e_1) = 1, c(e_2) = \varphi = \frac{\sqrt{5}-1}{2}, c(e_3) = 1$, and the capacity of all other edges is some really large constant (like $3 \cdot 10^8$.)

Now, run Ford-Fulkerson on this graph, but pick the following specific augmenting paths:

(a) At the very start, augment on the path $(s, v_2), (v_2, v_3), (v_3, t)$.
(b) Now, augment on path B.
(c) Now, augment on path C.
(d) Now, augment on path B.
(e) Now, augment on path A.
(f) Repeat, starting at step (b).

Show that all of these paths are augmenting, and thus that Ford-Fulkerson could conceivably pick these paths; show that the flow converges to a flow with value $4+\sqrt{5}$, and finally notice that this is far far less than its maximum possible value (which is $2 \cdot 3 \cdot 10^8 + 1$.)

2. (−) Suppose we do the following:

   - Let G be an abelian group of order $2n-1$ – say, the integers mod $2n-1$.
   - For each $g \in G$, let $M_g = \{\{g, \infty\} \cup \{a, b\} : a + b = 2g, a \neq b\}$.
   - Identify each element in the set $G \cup \{\infty\}$ with a team, and each of the $2n-1$ rounds with the $2n-1$ distinct M_g's.

   Show that we've created a tournament where each team plays each other team exactly once.

3. Recall the definitions of A-circulation and A-closure, where A is an abelian group (most often, the integers modulo some value k:)

   Definition. $(*)$ A directed graph G has an A-**circulation** g iff there is some way to assign elements of A to every edge in G, such that we obey Kirchoff's law at every vertex v in G:

   $$\sum_{w \in N^+(v)} g(v, w) - \sum_{w \in N^-(v)} g(w, v) = 0.$$

   We say that an A-circulation is in fact an A-**flow** iff $g(e) \neq 0$, for every edge $e \in E(G)$.

   Show that for any graph G, there is a polynomial P such that the number of distinct A-flows on G is given by $p(n-1)$, where $|A| = n$. (Hint: work by induction on $|E|$. Take any edge e_0 in your graph, and consider the two graphs $G \setminus \{e_0\}$, where you just delete the edge, and $G/\{e_0\}$, where you shrink the edge to a point. Apply your inductive hypothesis to these two sets. How can you combine their polynomials?

4. (−) Use the above to prove that the only relevant thing about groups in A-flows is their size: i.e. if a group A has an A-flow on some graph G, then so does any other group with the same cardinality.