

## Homework 4

Week 2

Mathcamp 2011

Attempt the problems that seem interesting! Easier exercises are marked with  $(-)$  signs; harder ones are marked by  $(*)$ . Open questions are denoted by writing  $(**)$ , as they are presumably quite hard. Oh! Also, typos build character: if you find any (not that there ever could be such things in my problem sets,) correct them to the most reasonable thing you can think of and proceed from there!

1. Hello!
2. Prove the many many many things mentioned in lecture today! Specifically:
  - (a)  $(-)$  Show that a connected graph  $G$  has  $\varphi(G) = \infty$  whenever  $G$  has a **bridge**: i.e. there is an edge  $e \in E(G)$  such that removing  $e$  from  $G$  disconnects  $G$ .
  - (b)  $\varphi(K_4) = 4$ .
  - (c)  $(*)$  If  $n$  is even and not equal to 2 or 4, then  $\varphi(K_n) = 4$ .
  - (d) A graph  $G$  has a 4-flow if and only if we can write it as the **union** of two even graphs. (Hint: use the Klein-4 group!)
  - (e) A cubic graph  $G$  has a 4-flow if and only if it is three-edge colorable.
  - (f)  $(**)$  Every bridgeless graph has a 5-flow.
  - (g)  $(**)$  Every bridgeless graph without the Petersen graph as a minor has a 4-flow.
3.  $(-)$  Find a graph with the Petersen graph as a minor that has a 2-flow: conclude that Tutte's conjecture is not an if and only if statement.
4. Consider the following application of flows to **open-pit mining**: suppose you have a  $x \times y$  plain you're going to mine for minerals. Suppose further that you've surveyed, via samples extracted from the earth, the ground underneath your plain to a depth  $z$ : by combining this data, you've created a  $x \times y \times z$  lattice, each point of which corresponds to some chunk of earth you know something about.

Specifically, assume that to each chunk of earth you've assigned some value  $b_i$ , which can be either positive or negative (i.e.  $b_i$  is the value of the ore in that chunk of earth, minus the cost of excavating and processing it.) Assume that you cannot dig out a chunk of earth without removing the chunk of earth directly above it, at the least. How can you use a system of flows to quickly determine which chunks of earth you should mine to maximize your profits?