| Introduction to Graph Theory | Instructor: Padraic Bartlett |
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# Lecture 4: The Chromatic Number

*Week 1* *Mathcamp 2011*

In our discussion of bipartite graphs, we mentioned that one way to classify bipartite graphs was to think of them as graphs that are **2-colorable**: i.e. graphs in which we could color all of the vertices either red or blue, so that no edge would have two endpoints of the same color. When phrased in this way, an obvious question we could ask is the following: Is there a good way to generalize this idea of 2-coloring to $k$-coloring? Does anything interesting result from such a definition?

As it turns out: yes!

## 1 Basic Definitions

**Definition.** We say that a graph $G$ is $k$**-colorable** if we can assign the colors[1] $\{1, \ldots k\}$ to the vertices in $V(G)$, in such a way that every vertex gets exactly one color and no edge in $E(G)$ has both of its endpoints colored the same color. We call such a coloring a **proper coloring**, though sometimes where it's clear what we mean we'll just call it a coloring.

Alternately, such graphs are sometimes called $k$-partite. For a fixed graph $G$, if $k$ is the smallest number such that $G$ admits a $k$-coloring, we say that the chromatic number of $G$ is $k$, and write $\chi(G) = k$.

To illustrate how this definitions goes, we work a few examples:

1. $K_n$: The complete graph on $n$ vertices has chromatic number $n$. To see that it is at least $n$, simply paint each of the vertices $\{v_1, \ldots v_n\}$ of $V(K_n)$ a different color (say, $v_i$ is painted $i$;) then every edge trivially has two endpoints of different colors. To see that this is necessary, take any proper coloring of $K_n$, and look at any vertex $v_i$: because it's connected to every other vertex, it cannot be the same color as any other vertex (and therefore must have a different color than every other vertex, which forces $n$ colors.)

2. Edgeless graphs: If a graph $G$ has no edges, its chromatic number is 1; just color every vertex the same color. These are also the **only** graphs with chromatic number 1; any graph with an edge needs at least two colors to properly color it, as both endpoints of that edge cannot be the same color.

3. Bipartite graphs: By definition, every bipartite graph with at least one edge has chromatic number 2.

4. The pentagon: The pentagon is an odd cycle, which we showed was not bipartite; so its chromatic number must be greater than 2. In fact, its chromatic number is 3:

[1] By "color," we just mean a collection of distinct labels, like (say) natural numbers. Actual colors have the disadvantage of being finite in number, which is rather pesky.

simply color its vertices $R, G, R, G, B$ in order by walking around the perimeter of the pentagon. (In fact, this same idea can be used to show that any cycle of length $2k + 1$ is 3-colorable: we know that these are not bipartite, and that they do admit 3-colorings via the $R, G, R, G \ldots R, G, B$-coloring described above.)

5. In our first lecture, we said that one way of phrasing the 4-color theorem was to say that all "map-graphs" could be colored with at most four colors. In the language we've described above, this is the claim that all "map-graphs" have chromatic number $\leq 4$.

## 2 Properties and Examples

We developed this notion of $k$-chromatic graphs by generalizing the concept of bipartite graphs. A natural question to ask, then, is whether our earlier classifcation of bipartite graphs can be generalized to $k$-partite graphs. I.e.: we showed that a graph was bipartite if and only if it didn't contain any odd cycles. Is there a similar classification for all graphs with chromatic number (say) 3?

Surprisingly: no! While there certainly are tons of 3-chromatic graphs, there is no materially different classification of all of them beyond "there is a 3-coloring of this graph" that graph theorists have found. In fact, while graph theorists have been studying colorings pretty much since the 4-color theorem was postulated, there really is a lot that we don't know out there! (For example: consider the **unit-distance graph**, which has vertex set $\mathbb{R}^2$ and an edge between two points in the plane if and only if the distance between them is 1. In the HW, you'll prove that it's chromatic number is between 4 and 7: to this day, these are the best known bounds.)

However, we can say a few things about how the chromatic number relates to some other properties of a graph. We state a few relevant definitions below, and then prove a few related propositions:

**Definition.** For a graph $G$ and a subgraph $H$, we say that $H$ is a **induced subgraph** of $G$ if and only if whenever $u, v \in V(H)$ and $\{u, v\} \in E(G)$, we have that $\{u, v\} \in E(H)$. In other words, $H$ is a subgraph made by picking out some vertices from within $H$, and then adding in **every** edge in $G$ that connects those vertices.

**Definition.** For a graph $G$, we define the **clique number** of $G$, $\omega(G)$, to be the largest value of $k$ for which $K_k$ is an induced subgraph of $G$. As every nonempty graph contains (at the minimum) a $K_1$ as an induced subgraph, this is a well-defined quantity.

**Proposition 1** *If $G$ is a graph and $H$ is any subgraph of $G$, $\chi(G) \geq \chi(H)$.*

**Proof.** This is remarkably trivial. If $G$ admits a $k$-coloring, then simply take some proper $k$-coloring of $G$ and use it to color $H$'s vertices. Because $H$'s edges are all in $E$, we know that none of these edges are monochromatic under this coloring; therefore, it is a proper $k$-coloring of $H$, and thus $\chi(H) \leq k = \chi(G)$.

**Proposition 2** *If $G$ is a graph, $\chi(G) \geq \omega(G)$.*

**Proof.** Let $H$ be an induced subgraph of $G$ isomorphic to $K_{\omega(G)}$, which exists by definition. Then, by the above proposition, $\chi(G) \geq \chi(K_{\omega(G)}) = \omega(G)$.

This gives us a lower bound. The following definition and proposition give us an upper bound, as well:

**Definition.** For a graph $G$, let $\Delta(G)$ denote the maximum degree of any of $G$'s vertices, and $\delta(G)$ denote the minimum degree of any of $G$'s vertices.

**Proposition 3** *For any graph $G$, $\chi(G) \leq \Delta(G) + 1$.*

**Proof.** The algorithm here is remarkably simple, but at the same time important enough that we give it a name: the **greedy algorithm**. We define it here:

- (Greedy algorithm.) As input: take in a graph $G$ with vertex set $V(G) = \{v_1, \dots v_n\}$, and a list of potential colors $\mathbb{N}$.
- At stage $k$: look at $v_k$, and color it the smallest color in $\mathbb{N}$ not yet used on any of $v_k$'s neighbors.

By construction, this creates a proper coloring of $G$. As well, because each vertex has $\leq \Delta(G)$ neighbors, we'll always have at least one choice of a color that's less than $\Delta(G)+1$; therefore, this creates a proper coloring of $G$ that uses $\leq \Delta(G) + 1$ colors! So $\chi(G) \leq \Delta(G) + 1$, as claimed.

To sum up: we've shown that for any graph $G$, we have

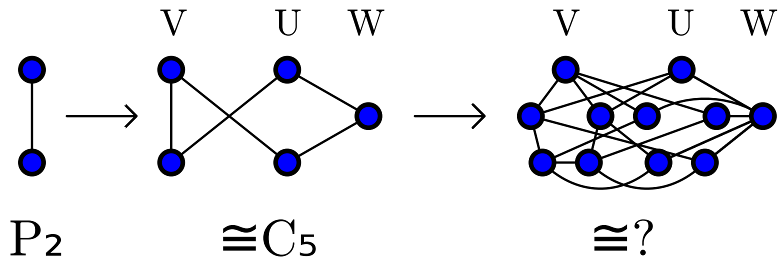
$$\omega(G) \leq \chi(G) \leq \Delta(G) + 1.$$

Which is something! However, as it turns out, it's not a *lot*; in general, the gap between $\omega(G)$ and $\Delta(G)$ can be quite large, as the following family of graphs shows:

**Example.** The **Mycielski** construction is a method for turning a triangle-free graph with chromatic number $k$ into a larger triangle-free graph with chromatic number $k+1$. It works as follows:

- As input, take a triangle-free graph $G$ with $\chi(G) = k$ and vertex set $\{v_1, \dots v_n\}$.
- Form the graph $G'$ as follows: let $V(G') = \{v_1, \dots v_n\} \cup \{u_1, \dots u_n\} \cup \{w\}$.
- Start with $E(G') = E(G)$.
- For every $u_i$, add edges from $u_i$ to all of $v_i$'s neighbors.
- Finally, attach an edge from $w$ to every vertex $\{u_1, \dots u_n\}$.

Starting from the triangle-free 2-chromatic graph $K_2$, here are two consecutive applications of the above process:




**Proposition 4** *The above process does what it claims: i.e. given a triangle-free graph with chromatic number $k$, it returns a larger triangle-free graph with chromatic number $k+1$.*

**Proof.** Let $G, G'$ be as described above. For convenience, let's refer to $\{v_1, \ldots v_n\}$ as $V$ and $\{u_1, \ldots u_n\}$ as $U$. First, notice that there are no edges between any of the elements in $U$ in $G'$; therefore, any triangle could not involve two elements from $U$. Because $G$ was triangle-free, it also could not consist of three elements from $V$; finally, because $w$ is not connected to any elements in $V$, no triangle can involve $w$. So, if a triangle exists, it must consist of two elements $v_i, v_j$ in $V$ and an element $u_l$ in $U$; however, we know that $u_l$'s only neighbors in $V$ are the neighbors of $v_l$. Therefore, if $(v_i, v_j, u_l)$ was a triangle, $(v_i, v_j, v_l)$ would also be a triangle; but this would mean that $G$ contained a triangle, which contradicts our choice of $G$.

Therefore, $G'$ is triangle-free; it suffices to show that $G'$ has chromatic number $k+1$.

To create a proper $k+1$-coloring of $G'$: take a proper coloring $f : V(G) \to \{1, \ldots k\}$ and create a new coloring map $f' : V(G') \to \{1, \ldots k+1\}$ by setting

- $f'(v_i) = f(v_i)$,
- $f'(u_i) = f(v_i)$, and
- $f(w) = k+1$.

Because each $u_i$ is connected to all of $v_i$'s neighbors, none of which are colored $f(v_i)$, we know that no conflicts come up there; as well, because $f(w) = k+1$, no conflicts can arise there. So this is a proper coloring.

Now, take any $k$-coloring $g$ of $G'$: we seek to show that this coloring must be improper, which would prove that $G'$ is $k+1$-chromatic. To do this: first, assume without any loss of generality that $f(w) = k$ (it has to be colored something, so it might as well be $k$.)

Then, because $w$ is connected to all of $U$, the elements of $U$ must be colored with the elements $\{1, \ldots k-1\}$. Let $A = \{v_i \in V : g(v_i) = k\}$. We will now use $U$ to recolor these vertices with the colors $\{1, \ldots k-1\}$: if we can do this properly, then we will have created a $k-1$ proper coloring of $G$, a $k$-chromatic graph (and thus arrived at a contradiction.)

To do this recoloring: change $g$ on the elements of $A$ so that $g(v_i)$'s new color is $g(u_i)$. We now claim that $g$ is a proper $k-1$ coloring of $G$ itself. To see this: take any edge $\{v_i, v_j\}$ in $G$. If both of $v_i, v_j \notin A$, then we didn't change the coloring of $v_i$ and $v_j$; so this edge is still not monochromatic, because $g$ was a proper coloring of $G'$. If $v_i \in A$ and $v_j \notin A$, then $v_j$ is a neighbor of $v_i$ and thus (by construction) $u_i$ has an edge to $v_j$. But this means that $g(u_i) \neq g(v_j)$, because $g$ was a proper coloring of $G'$: therefore, this edge is also not monochromatic!

Because there are no edges between elements of $A$ (as they were all originally colored $k$ under $g$, and therefore there weren't any edges between them,) this covers all of the cases: so we've turned $g$ into a $k-1$ coloring of a $k$-chromatic graph. As this is impossible, we can conclude that $g$ cannot exist – i.e. $G'$ cannot be $k$-colored! So $\chi(G') = k + 1$, as claimed.

As the example above illustrates, our bounds can (unfortunately) be rather loose: the Mycielskians, for example, have $\omega(M) = 2$ (because they don't even contain a triangle, $K_3$!), and yet have arbitrarily high chromatic number. Conversely, the complete bipartite graphs $K_{n,n}$ all have chromatic number 2, and yet have $\Delta(G) = n$; so our upper bound of $\Delta(G) + 1$ can also be rather misleading!