Introduction to Graph Theory	Instructor: Padraic Bartlett

Lecture 5: Ramsey Theory

Week 1 of 1 | *Mathcamp 2011*

In the last lecture, we introduced the idea of **graph colorings**, where we described how

The motivating question for this final lecture is the following:

Question 1 *Amongst any collection of 6 people, can you always find three mutual friends or three mutual strangers?*

Solution. Translating this into the language of graph theory, our question is the following: if you color the edges of K_6 red and blue, do you always have to create a triangle with monochromatically red or monochromatically blue edges?

We claim that you will always do so. To see why: pick any red-blue coloring of K_6, and any vertex $v \in K_6$. Because $\deg(v) = 5$, we know that if the five edges are shaded red and blue, there must be at least three of these edges that are the same color! Suppose (without any loss of generality) that this color is red, and let $\{w_1, w_2, w_3\}$ be the endpoints of these edges.

Then, there are two cases:

- There is some edge $\{w_i, w_j\}$ that's red. In this case, the vertices v, w_i, w_j form a red triangle.
- Every edge $\{w_i, w_j\}$ is blue. In this case, the vertices w_1, w_2, w_3 form a blue triangle.

In either situation, we've found a monochrome triangle! So these always exist.

As mathematicians, whenever we prove something we're really tempted to see if a generalization of it might be true. For example: in the above question, we showed that any two-coloring of K_6 creates a monochrome triangle (i.e. a K_3.) One natural question we could ask, then, is the following: given a fixed value of k, what values of K_n will a two-coloring of K_n's edges always force a K_k with monochrome edges to exist? Do such values of n even exist? Also, what if we use more than two colors – say, c colors? Will there still be values of n such that a c-coloring of K_n's edges always creates a monochrome K_k?

As it turns out, the answer to this question is yes! The result is called Ramsey's theorem:

Question 2 *(Ramsey's Theorem) Let $C = \{1, \dots c\}$ be a collection of c distinct colors, and $n_1, \dots n_c$ a collection of integers. Then, there is some value of n such that if K_n's edges are colored with c colors, it must contain a K_{n_i} where all of K_{n_i}'s edges are colored i, for some $1 \leq i \leq c$.*

Proof. Let $R(n_1, \dots, n_c)$ denote the smallest value of n such that if K_n's edges are colored with c colors, then K_n necessarily contains a i-monochrome K_{n_i} (i.e. a K_{n_i} where all of the edges are colored i.) We seek to show that R is well-defined, and always exists.

To do this, we proceed by induction on the number of colors $C = \{1, \ldots, c\}$. When $c = 1$, note that this is trivial, as $R(k) = k$ for all k.

Now, consider the two-color case. We have $R(n, 1) = R(1, n) = 1$, as any two-coloring of K_n's edges has a K_1 in which all of the edges are whatever color we want (because there are no edges in K_1.)

As well, we have $R(n, 2) = R(2, n) = n$, because any red-blue two-coloring of K_n's edges either

- paints all of the edges the same color (which makes a monochrome K_n of some color), or
- paints at least one edge red and another blue (which makes monochrome K_2's of both colors.)

Furthermore, we claim that we have the following recursive bound on the growth of $R(r, s)$:

$$R(r, s) \leq R(r, s-1) + R(r-1, s)$$

To prove this, we proceed by induction on the sum $r + s$. We've already proven the base cases via the two examples above: so we take any pair r, s, and can assume that our bound holds for any x, y with $x + y < r + s$.

Take a complete graph K on $(R(r, s-1) + R(r-1, s))$ many vertices, and color its edges red and blue (or 1 and 2, if you prefer integers). We seek to show that there's either a monochrome red K_r or monochrome blue K_s in K_n.

To see this: pick any $v \in K$, and partition the rest of K's vertices into two sets:

- B', which contains all of the vertices in K connected to v by a blue edge, and
- R', which contains all of the vertices in K connected to v by a red edge.

Let B and R be the subgraphs of K induced by these vertices, respectively.

Because K has

$$R(r, s-1) + R(r-1, s) = |V(B)| + |V(R)| + 1$$

many vertices, either $|V(B)| \geq R(r, s-1)$ or $|V(R)| \geq R(r-1, s)$.

Suppose that we have $|V(B)| \geq R(r, s-1)$. Because $r + s - 1 < r + s$, we can apply our inductive hypothesis, which tells us that we have either

1. a red K_r inside of B, or
2. a blue K_{s-1} inside of B, in which case (by combining this blue K_{s-1} with v and its edges to B) we have a blue K_s inside of our entire K_n.

These are the two cases we were looking for; so, in the situation where $|V(B)| \geq R(r, s-1)$, we've proven our claim!

Similarly, if we have $|R| \geq R(r-1, s)$, we can use induction to tell us that there's either

1. a blue K_s inside of R, or

2. a red K_{r-1} inside of R, in which case (by combining this red K_{r-1} with v and its edges to R) we have a red K_r inside of our entire K_n,

and we're also done.

Now, suppose that we've settled our theorem for $c-1$ colors. We seek to resolve it for c colors. Specifically, we make the following claim:

$$R(n_1, \ldots n_c) \leq R(n_1, \ldots n_{c-2}, (R(n_{c-1}, R_c))).$$

To see this: let K be the complete graph on $R(n_1, \ldots n_{c-2}, (R(n_{c-1}, R_c)))$ many vertices, and color K's vertices with c different colors.

Now: become selectively colorblind! In other words, pretend temporarily that $c-1$ and c have the same colors.

Then, by our inductive hypothesis, either

- there is an i-monochrome K_{n_i}, for some $1 \leq i \leq c-2$, or
- there is a $(c\&c-1)$-colored $K_{R(n_{c-1}, n_c)}$. By the definition of $R(n_{c-1}, n_c)$, this means that there's either a $(c-1)$-monochrome $K_{n_{c-1}}$, or a c-monochrome K_{n_c}.

So we're done!

In the language of the proof above, the opening question for this lecture can be thought of as showing $R(3,3) = 6$.

In general, Ramsey numbers are ridiculously hard to find. Paul Erdős, a famous combinatorialist and mathematician, was fond of telling the following story about finding something as simple as $R(6,6)$:

> Erdős asks us to imagine an alien force, vastly more powerful than us, landing on Earth and demanding the value of R(5, 5) or they will destroy our planet. In that case, he claims, we should marshal all our computers and all our mathematicians and attempt to find the value. But suppose, instead, that they ask for R(6, 6). In that case, he believes, we should attempt to destroy the aliens.

To illustrate some of the difficulty of finding such numbers, consider the following question:

Question 3 *What's* $R(3,4)$*?*

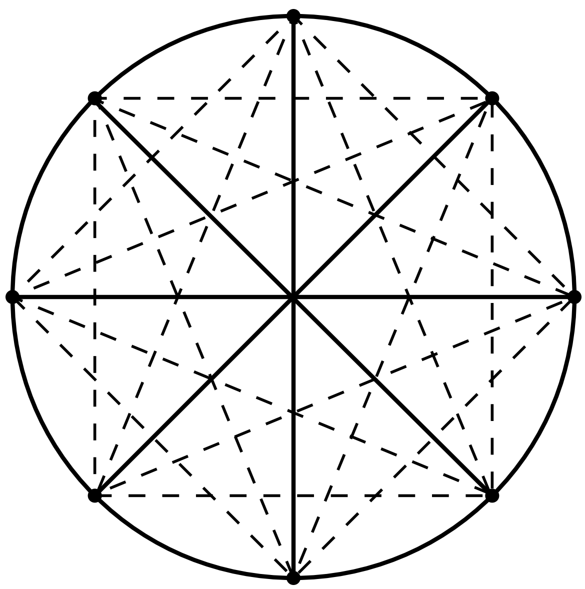
Solution. Pick n such that for any red-blue coloring of K_n, we have neither a blue K_3 nor a red K_4. Pick any $x \in K_n$, and again let

- B be the subgraph induced by the set of vertices in K_n connected to v by a blue edge, and
- R be the subgraph induced by the set in K_n connected to v by a red edge.

If there is a blue edge in B, then $x \cup B$ will yield a blue K_3; similarly, if there is a red K_3 in R, $x \cup R$ yields a red K_4. Because $R(2,4) = 4$ and $R(3,3) = 6$, we have that if neither situation occurs, we must have $|B| \leq 3$ and $|R| \leq 5$. In other words, we've just shown that for any vertex $x \in K_n$, we have $deg_b(x) \leq 3$ and $deg_r(x) \leq 5$. Consequently, the total degree of x must be ≤ 8; i.e. $n \leq 9$, and thus $R(3,4) \leq 10$.

Consider the case $n = 9$. In this case, each x must have $deg_b(x) = 3$ and $deg_r(x) = 5$; consequently, the number of blue edges in K_n can be counted, via the degree-sum formula, to be $\frac{1}{2}\sum_{x \in K_n} deg_b(x) = 27/2 = 13.5$. Since we can't have half of a blue edge, this is also impossible! So $R(3,4) \leq 9$.

Conversely: consider the following drawing below. The solid edges form a graph with girth ≥ 4, and so do not contain a K_3. As well, picking any four points on the boundary of a 8-cycle necessarily involves picking two opposite points or two adjacent points; so there is no complete K_4 amongst 4 points within the dashed edges.



Thus, $R(3,4) > 8$; i.e. $R(3,4) = 9$.