Introduction to Graph Theory	Instructor: Padraic Bartlett
Lecture 6: Snarks!	
Week 1 of 1	*Mathcamp 2011*

1 Snarks!

Theorem 1 *(Snark Theorem: 2001, Robertson, Sanders, Seymour, and Thomas) Every snark contains the Petersen graph as a minor.*

Corollary 2 *The four-color theorem holds.*

Without knowing what a snark even **is**, the above pair of results should hopefully motivate just **why** they're such fascinating things to study – given the snark theorem, it's remarkably easy to prove the four-color theorem (indeed, it's on your HW!) Today's lecture is going to be a brief introduction to just what snarks **are**: in the next hour, we will define snarks, and go about the remarkably strange process of hunting them...

First, we make a series of definitions:

Definition. The **line graph** $L(G)$ of a graph G is the graph with vertex set given by the edges of G, and an edge $\{e, f\}$ in G if and only if these two edges are incident in G. A n**-edge coloring** of a graph G is a mapping from the set $E(G)$ into the set $\{1, 2, \ldots n\}$ such that no two incident edges receive the same colors. The **edge chromatic number** of a graph G, $\chi'(G)$, is the smallest value of n such that G admits a n-edge coloring.

To give a feel for how these definitions work, we study a few quick examples:

Proposition 3 *A cycle C_n has edge-chromatic number $\chi'(G) = \chi(G)$.*

Proof. Take a cycle C_n, and consider its line graph $L(C_n)$. This is another cycle! In fact, it's the same cycle as G, as it has the same number of vertices; thus, its edge chromatic number is the same as G.

Theorem 4 *If $G = (A, B)$ is a bipartite graph, then $\chi'(G) = \Delta(G)$.*

Proof. On the HW!

With this out of the way, we can now define a snark!

Definition. A snark is a graph G with the following properties:

1. G is connected.
2. G is **3-regular**: i.e. every vertex in G has degree 3.
3. G is **bridgeless**; i.e. if we remove any one edge from G, the resulting graph is still connected.

4. G has girth ≥ 5: i.e. it has no subgraphs isomorphic to cycles of length ≤ 4.
5. $\chi'(G) \geq 4$.

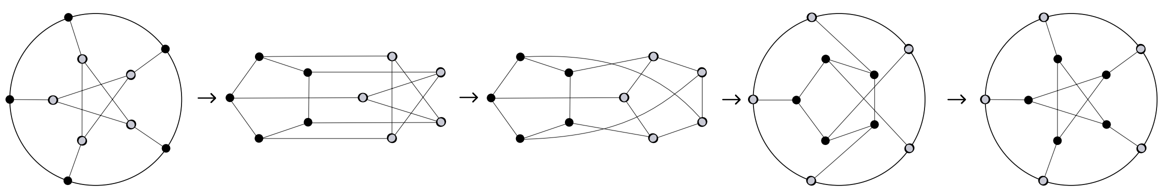
This might seem like a somewhat…odd definition. Why did we make it? Well: as it turns out, the definition of a snark arose from an attempt to generalize the **Petersen**graph to a family of graphs. We prove that the Petersen graph is indeed a snark here:

Proposition 5 *The Petersen graph P is a snark.*

Proof. We first note the following useful lemma:

Lemma 6 *There is an automorphism of the Petersen graph that swaps the outer pentagon and the inner star.*

Proof. In this case, a picture is worth a thousand proofs:



Given the above lemma, we now proceed to check the five properties required to be a snark:

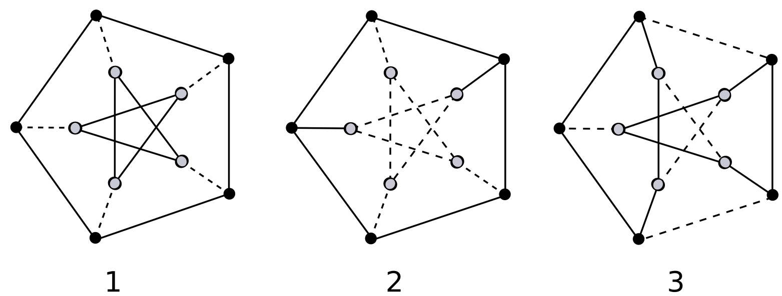
- Connected: trivially true.
- Bridgeless: also trivially true.
- 3-regular: again, trivially true, as every vertex has degree 3.
- Girth 5: suppose not, that P has a cycle of length ≤ 4. Such a cycle cannot live entirely within the inner or outer 5-cycles of P; so it has to involve two of the "cross-edges" (the edges connecting the outer pentagon and inner star) of P. Pick any two such cross-edges; then, by our lemma, we can insist (by moving P around) that these cross-edges involve two non-adjacent vertices on the outer cycle of P. But then we have to use at least two more edges on the outer cycle to connect these two cross-edges! So this cycle must have ≥ 5 edges.
- 4-edge-colorable: to see this, again proceed by contradiction. Suppose not; that we have a way of partitioning P's edges into 3 color classes, R, G, and B in such a way that within each color class, there are no two adjacent edges. Then each color class can have no more than $|V(P)|/2 = 10/2 = 5$-many edges, as we can use each vertex at most once in a given color class and each edge uses two vertices. But $|E(P)| = 15$ – so each color class has exactly 5 edges! In other words, each color class is a 1-factor[1]!

  We seek to show that this is impossible: i.e. that P cannot be decomposed into 1-factors. So: to do this, take any 1-factor and delete it from P. We then claim

[1] A **1-factor** of a graph G is a subgraph made of disjoint edges that hits every vertex in G.

that the resulting 2-factor is isomorphic to a pair of disjoint pentagons, and thus cannot be decomposed into 2 1-factors (as doing so would create a 2-edge-coloring of a pentagon.)

First, observe that in any 2-factor, we always have an even number of cross-edges. Why is this? Because 2-factors are made out of disjoint cycles: thus, if any cycle leaves either the inside or outside along a cross-edge, it must return along another cross-edge. So, three possibilities exist:



- We use no cross-edges. In this case, we have two pentagons; specifically, the inner and outer pentagons of P.
- We use 2 cross-edges. In this case, we can again insist (by our lemma) that the cross-edges used are specifically the two depicted above. In this case, because these two cross edges involve nonadjacent endpoints, they force us to include the entire outer cycle of P in our 2-factor – but this creates vertices of degree 3! So this is impossible.
- We use 4 cross-edges. In this case, the cycle edges forced into our 2-factor again form 2 pentagons.

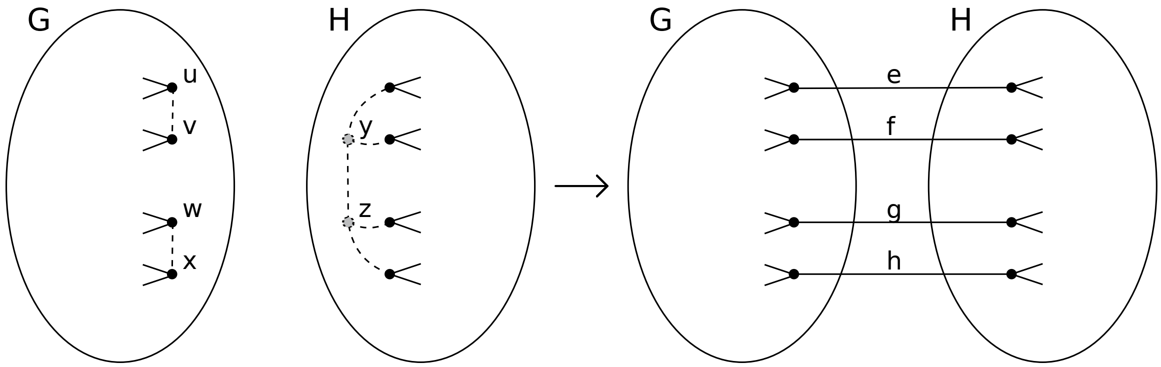
Snarks are a particular kind of graph that have been intensely studied since the 1880's, when Tait showed that proving the Snark Theorem would imply the four-color theorem; their (rather curious) name stems from the Lewis Carrol poem "The Hunting of the Snark[2]." To this day, they remain a remarkably mysterious collection of graphs, about which modern graph theory knows rather little – indeed, by 1973, graph theoreticians had only discovered 5 snarks in total! In this last part of this lecture, we'll show how we can use a rather simple operation to create an infinte family of snarks.

Specifically: consider the **dot product**, an operation we define here:

Definition. Given a pair of snarks G, H, we can form their dot product by manipulating a pair of disjoint edges $\{u, v\}, \{w, x\}$ in G and adjacent vertices y, z in H as shown below:

[2]An exerpt from the poem:

They sought it with thimbles, they sought it with care;
They pursued it with forks and hope;
They threatened its life with a railway-share;
They charmed it with smiles and soap.



Proposition 7 *The dot product preserves snarkiness.*

Proof. We first claim that the only interesting property to check is whether the dot product of two snarks is a snark; if you're not persuaded that this is true, check the other properties yourself!

So: we first prove the following extremely handy lemma:

Lemma 8 *Suppose that G is a 3-regular graph that's 3-edge-colorable. Let Z be a collection of nonadjacent edges in G that satisfies the following property: if we delete the Z-edges from our graph G, G is disconnected into two components A and B, such that each edge of Z has one endpoint in A and one in B. Let n_i be the number of edges in Z colored i, for $i = 1, 2, 3$. Then the n_i are all congruent modulo 2.*

Proof. Let A and B be two parts of G that Z divides G into. Pick some color c_i, and look at the vertices of A. Because G is cubic, every vertex $a \in A$ has an edge of every color entering it; so there are two possibilities: either

- the c_i-colored edge entering a is in Z, or
- the c_i-colored edge entering a goes to some other vertex in A.

Consequently, we have that $|A|$ is equal to n_i plus some even number; as a result, all of the n_i's are congruent to $|A|$ (and thus to each other!) mod 2.

Revisit the dot product picture. Suppose, for contradiction, that this graph is 3-edge colorable, and fix some 3-edge-coloring. By our above lemma, we know that all of the colors involved in $\{e, f, g, h\}$ have to be congruent mod 2; consequently, one color has to be omitted! Thus, we can say without loss of generality that the four edges above possess one of the following colorings:

- e, f, g, h are all colored 1;
- e, f are colored 1, g, h are colored 2;
- e, g are colored 1, f, h are colored 2.

In case 1, we can turn this into a 3-edge-coloring of G by coloring both u, v and w, x 1; in case 2, we can color the five edges deleted when we removed y and z 1, 2,3 as depicted below; and in case 3, we can just color u, v 1 and w, x 2. So we're done!