Introduction to Graph Theory	Instructor: Padraic Bartlett

Homework 1

Week 1 — *Mathcamp 2011*

1. A sequence $d_1 \geq d_2 \geq \ldots d_n$ of nonnegative integers is called **graphic** if and only if there is a graph G on n vertices such that $\deg(v_i) = d_i$, for every $v_i \in V(G)$.

   Determine whether any of the following sequences are graphic:

   - $5, 3, 3, 2, 2, 2$.
   - $6, 2, 2, 2$.
   - $3, 2, 2, 2, 1, 1, 1$
   - $3, 3, 3, 3, 3, 3, 3, 3, 3, 3$
   - $n, n, n \ldots n$.
   - $n, n, \ldots n, m, m \ldots m$

2. $(*)$ Prove the following theorem of Havel and Hakimi on graphic sequences: A sequence

   $$s, t_1, t_2, \ldots t_s, d_1, \ldots d_n$$

   is graphic if and only if the sequence

   $$t_1 - 1, t_2 - 1 \ldots t_s - 1, d_1, \ldots d_n$$

   is graphic.

3. $(-)$ Show that if G is a connected graph, then for any two vertices $x, y \in V(G)$, there is a path from x to y that doesn't repeat any vertices.

4. How many distinct graphs G are isomorphic to K_n, but are not equal to K_n?

5. $(*)$ How many distinct graphs G are isomorphic to the Petersen graph, but are not equal to K_n?

6. $(-)$ Show that if G is a graph with n vertices and m edges, then $n \leq m+1$. Similarly, show that any graph with n vertices and m edges has at least $n - m$ connected components.

7. $(-)$ In a graph G, we say that an edge e is a **cut-edge** if removing e from our graph increases the number of connected components in G. Show that if e is an edge, there is no subgraph of G that contains e and is isomorphic to a cycle.

8. $(*)$ A graph G is called **Eulerian** if it contains a path P that satisfies the following two properties:

   - P starts and ends on the same vertex.

- P uses every edge in G exactly once.

Show that a graph G is Eulerian if and only if the degree of every vertex in G is even.

9. $(-)$ Suppose that G is a graph and k is a positive integer ≥ 2 such that $\deg(v) \geq k$, for every $v \in V(G)$. Then G contains a cycle of length $k+1$.