1. Prove that for a graph G on n vertices, the following statements are equivalent:

   (a) G is a tree.
   (b) G is connected and has $n - 1$ edges.
   (c) G has $n - 1$ edges and no cycles.
   (d) For every pair u, v in $V(G)$, there is exactly one path from u to v that doesn't repeat any vertices.

2. Show that any connected graph on n vertices with m edges has at least $m - n + 1$ cycles.

3. $(-)$ A **spanning tree** T in a graph G is a subgraph T of G that is isomorphic to a tree, and also contains every vertex in $V(G)$. Show that every graph G contains a spanning tree.

4. $(-)$ How many spanning trees does K_n contain?

5. Show that if G is a graph where every vertex has degree $\geq k$ and T is a tree with k edges, then G contains T as a subgraph.

6. $(-)$ Find a graph G and tree T where every vertex in G has degree $\geq k - 1$, T has k edges, but G does not contain T as as subgraph.

7. A *graceful labeling* of a graph with E edges is a labeling $l(v)$ of its vertices with distinct integers from the set $\{0 \ldots E\}$, such that each edge $\{u, v\}$ is uniquely determined by the difference $|l(u) - l(v)|$. Show that all path-trees[1] are graceful.

8. Show that K_n is graceful if and only if $n \leq 4$.

9. $(*)$ A *caterpillar tree* is a tree such that deleting all of its leaves leaves us with a single path. Show that all caterpillar graphs are graceful.

10. $(**)$ Show that all trees are graceful.

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[1]A path-tree is a tree that consists of a single path.