Introduction to Graph Theory	Instructor: Padraic Bartlett

Homework 3

Week 1 *Mathcamp 2011*

1. ($-$) A graph G is called **k-critical** if $\chi(G) = k$. Show that every k-chromatic graph has a k-critical subgraph.

2. ($-$) Show that every k-critical graph is connected.

3. If G is k-critical, then the degree of every vertex in G is at least $k-1$.

4. ($-$) Show that if G is a k-critical graph, then $k \cdot (|V(G)| - 1) \leq 2 \cdot |E(G)|$.

5. Let G be a graph such that $\chi(G \setminus \{x, y\}) = \chi(G) - 2$, for all vertices in G. Show that G must be the complete graph.

6. ($**$) Suppose that G is a graph such that $\chi(G \setminus \{x, y\}) = \chi(G) - 2$, for all pairs of **adjacent** vertices in G. Show that G must be the complete graph. (This has been resolved for $k \leq 5$, and is open for $k = 6$ and higher, though some partial results are known. To make this a solvable problem, simply prove the question for $k \leq 5$. Perhaps relevantly, these graphs are called **double-critical graphs**.)

7. Given a collection $I\{I_1, \ldots I_n\}$ of intervals on the real line, define the **interval graph** G_I on the vertex set $\{v_1, \ldots v_n\}$ by drawing an edge $\{v_i, v_j\}$ if and only if $I_i \cap I_j \neq \emptyset$. Show that any interval graph has $\chi(G) = \omega(G)$.

8. Prove that if G is a graph, then $\chi(G) \leq 1 + \max_{i=1}^{n} \left(\min\{\deg(v_i), i-1\}\right)$.

9. ($*$) (Brook's theorem:) If G is a graph that's neither a complete graph nor an odd cycle, then $\chi(G) \leq \Delta(G)$. (Hint: First, prove this in the case that G has a vertex v with $\deg(v) < \Delta(G)$, by finding an appropriate spanning tree of G and applying a greedy coloring. Then, consider the case where G has all of its vertices of degree k; how can you extend our earlier idea to work in this situation?)