Introduction to Graph Theory	Instructor: Padraic Bartlett

Homework 5

Week 1	*Mathcamp 2011*

1. Show that if a graph G has chromatic number k, then it must have at least $\binom{k}{2}$ edges.
2. ($*$) What graphs on n vertices with chromatic number $\leq r$ has the most edges? Is there a unique such graph? (Hint: consider the complete r-partite graphs[1], where each part has size $\sim n/r$. How can you generalize these graphs when n does not divide r?)
3. ($*$) Suppose you have n fireflies on a sidewalk, such that no two fireflies are more than one foot away from each other. Show that if $n \geq 4$, there is at least one pair of fireflies within $1/\sqrt{2}$ of each other. (You can, in fact, show much more: there are at most $\lfloor n^2/3 \rfloor$ pairs of vertices that are not within $1/\sqrt{2}$ of each other, out of the $n(n-1)/2$ total possible pairings.)
4. Prove that a bipartite graph has edge-chromatic number given by $\Delta(G)$.
5. ($-$) A graph is called **planar** if there is some way to draw it in the plane so that none of its edges overlap at spots that are not endpoints. Show that K_4 is planar (even though the normal way of drawing it has overlapping edges!)
6. Prove that the Petersen graph is not planar!
7. ($*$) The **four-color theorem**, in the language of graph theory, is the statement that any planar graph has chromatic number ≤ 4. Prove that the snark theorem implies the four-color theorem.

[1] The complete r-partite graph on vertex sets $V_1, \ldots V_r$ is the graph formed by connecting any two vertices whenever they come from different parts. These graphs are trivially r-colorable, and not colorable with any less than r colors as long as none of the V_i's are empty.