Martin's Axiom	Instructors: Susan Durst and Padraic Bartlett
Lecture 1: Martin's Axiom And Domination	
Week 5	*Mathcamp 2011*

1 Notation and Assumptions

Throughout this class, we'll be working in ZFC. As you saw yesterday, choice is your friend! Deal with it.

Secondly, a couple little notational things:

- We use $\aleph_\alpha$ and ω_α to denote the same set-theoretic object. We'll try to use ω_α when we mean to use it as an ordinal number, and $\aleph_\alpha$ when we mean to use it as a cardinal number. This will often not actually happen.
- If A and B are sets, I use ${}^A B$ to denote the set of functions from A to B. In particular, I use ${}^\omega\omega$ to denote the set of functions from ω to ω.

2 Dominating Functions

Martin's axiom, at first glance, is a perplexing thing. In today's lecture, we're going to tell the story of **why** we care about this axiom: something we will do by studying **dominating functions**!

Definition. Let f, g be a pair of functions in ${}^\omega\omega$. We say that f **dominates** g, and write $f <_* g$, if and only if there is some $n_0 \in \omega$ such that whenever $n > n_0$, we have

$$f(n) < g(n).$$

We say that a function g dominates an entire subset $\mathscr{F} \subseteq^\omega \omega$ iff g dominates every element $f \in \mathscr{F}$.

Suppose we have any collection of functions $\mathscr{F} \subseteq^\omega \omega$. Can we find a function g that dominates **every** function in $\mathscr{F}$?

Theorem 1 *No.*

Proof. If $\mathscr{F} =^\omega \omega$, this is clearly impossible: given any candidate for a dominating function g, the function $g' = g + 1$ is also in ${}^\omega\omega$, and therefore g cannot dominate this entire set.

Ok, fine. That was a dumb question to ask. Here's a better one: for what cardinalities of $\mathscr{F}$ can we always find a function g that dominates all of $\mathscr{F}$?

This might have better luck. Specifically, when $\mathscr{F}$ is finite, our task is trivial: if $\mathscr{F} = \{f_1, \ldots f_k\}$, just define $g(n) = \max\{f_1(n), \ldots f_k(n)\} + 1$.

How about when $\mathscr{F}$ is countable? Plausibly enough, we can find a dominating function for $\mathscr{F}$ in this case as well:

Theorem 2 *If $\mathscr{F}$ is a countable subset of ${}^\omega\omega$, then there is a function g that dominates all of $\mathscr{F}$.*

Proof. Let $\mathscr{F} = \{f_i\}_{i=1}^{\infty}$. For each n, define g as follows:

$$g(n) = \left(\max_{i \leq n} f_i(n)\right) + 1.$$

Then, for any $f_k \in \mathscr{F}$, g is strictly larger than f on all values of $n \geq k$; so g is a dominating function for $\mathscr{F}$!

Sweet: so, we've proven that we can do this for every countable set, and we've proven we can't do it for some sets of size $|{}^\omega\omega| = |\mathbb{R}|$. (If you don't believe this equality, good news! It's on the HW!)

Can we say anything else? Well: suppose the continuum hypothesis[1] holds! Then there are no intermediate sizes of infinity, and we can't say anything else. Sadface.

But! Suppose that the continuum hypothesis **doesn't** hold! Then there are uncountable sets $\mathscr{F}$ with cardinalities $|\omega_1| < |{}^\omega\omega|$. What can we say for these sets – do they, too, have dominating functions (and are thus in some sense "countable-ish?") Or are they also too big to have dominating functions?

3 Motivation: Re-Examining Countable Sets

To get a better idea of what's going on, let's look at what we did in the case where we succeeded in finding a dominating function: i.e. where $\mathscr{F}$ was countable. There, we **constructed** g one step at a time, as follows:

- At each step n, we chose one value for the function g.
- This isn't all that we did! At every step, we also made a "promise" about our function g's later behavior Specifically, we said that at every step past this n-th step, our function g will dominate the function f_n: i.e. we'll always pick values of g that exceed those of f_n from here on out.

The reason that countability is so nice for this process is that it allows us to proceed **linearly**: we can order our set of functions $\mathscr{F}$ like ω, and just proceed step-by-step! Basically, each step gives us more information about our function; then, by taking limits, this process will give us a beautiful fully-formed function to work with.

Unfortunately, when we have an uncountable number of functions to take care of, there's no way to proceed linearly: ω_1, tragically, cannot be ordered in such a way to look like ω. Whatever can we do?

[1] The continuum hypothesis is the assertion that the cardinality of the continuum is the cardinality of the first uncountable ordinal: i.e. that $|\omega_1| = |2^\omega|$. A famous result of Gödel and Cohen asserts that the truth of the continuum hypothesis cannot be determined from the axioms of ZFC: i.e. assuming that CH holds or does not hold cannot create any new contradictions in ZFC.

4 POOOOOOOOOOOOOOSET!

That's right: Posets. Specifically: suppose we're trying to build a function, but not in a linear fashion (as that seems out of the question. How can we do this? Well: we'd want to do something like the following

- At every step, we will have a finite piece of a function, which we're going to call φ.
- We also have a finite set of promises that we've made: i.e. a finite subset $\mathscr{F}_0 \subset \mathscr{F}$.
- When I proceed to the next step, I'm going to want an extension of my function φ which keeps all the promises that I made in $\mathscr{F}_0$, and maybe makes a few more.

How can we realize this? Via the poset $\mathbb{P}_{\mathscr{F}}$, which we construct here:

- Elements of $\mathbb{P}_{\mathscr{F}}$: pairs $(\varphi, \mathscr{F}_0)$, where φ is a function $\{1, \ldots n\} \to \omega$ and $\mathscr{F}_0$ is a collection of "promises" that φ will keep.
- What do we mean by "promises?" Well: one good interpretation to make is that for each f in $\mathscr{F}_0$, we promise that whenever we extend φ into a bigger piece of g, we will make sure that whatever values we add for $g(n)$ are bigger than the value of $f(n)$.

  This raises the question of what we mean by "extension." We define this by the ordering we have on our poset:
- We say that $(\psi, \mathscr{F}_1)$ is an extension of $(\varphi, \mathscr{F}_0)$, and define $(\varphi, \mathscr{F}_0) \geq (\psi, \mathscr{F}_1)$ in our poset, if the following things occur:
  1. ψ is an extension of φ: i.e. $\text{dom}(\psi) \supseteq \text{dom}(\varphi)$, and they agree at every value where φ is defined.
  2. $\mathscr{F}_0 \subseteq \mathscr{F}_1$.
  3. For any $f \in \mathscr{F}_0$, $m \in \text{dom}(\psi) \setminus \text{dom}(\varphi)$, we have $\psi(m) > f(m)$: i.e. ψ keeps the "promises" that φ made.

What does this give us? Well: it tells us that whenever we have two elements $(\varphi, \mathscr{F}_0) \geq (\psi, \mathscr{F}_1)$, ψ is in some sense a "more complete" function: i.e. it's closer to a complete function that φ, and is therefore more constrained.

This is certainly a ... thing. How can we use it to set about finding our dominating function g, or deciding if such a thing even exists?

Well: what if we **had** such a dominating function g? What would it tell us about our poset – would our poset have to satisfy certain properties?

By lecture theory[2], the answer here is yes!

Specifically: let G be the collection of ordered pairs $(\varphi, \mathscr{F}_0)$ such that

[2] For the literary kids out there: Chekov's Gun is a guideline for writing good plays coined by the Russian author/playwright Anton Chekhov. It's best described by the following quote of Chekhov: "If in the first act you have hung a pistol on the wall, then in the following one it should be fired. Otherwise don't put it there." Lecture theory is the application of Chekhov's Gun to mathematical lectures: if your lecturer has asked you a rhetorical question about whether some recently-introduced object might be useful, the answer is almost always yes. Unless they are trolling you. Which we would never do.

1. g is an extension of φ.

2. For any $f \in \mathscr{F}_0$, $m \in \omega \setminus \mathrm{dom}(\varphi)$, we have $g(m) > f(m)$.

In other words, G is the collection of all of the finite pieces of our dominating function g. Notice that G has the following nice properties:

1. If $p, q \in G$, then there exists $r \in G$ with $r \leq p, q$.

2. For all $p, q \in \mathbb{P}_{\mathscr{F}}$, if $p \leq q$ and $p \in G$, then $q \in G$.

3. For each $n \in \omega$, there exists $(\phi, \mathscr{F}_0) \in G$ such that $n \in \mathrm{dom}(\phi)$.

4. For each $f \in \mathscr{F}$, there exists $(\phi, \mathscr{F}_0) \in G$ such that $f \in \mathscr{F}_0$.

These properties are so nice, in fact, we're going to make them into **definitions**!

5 Filters and Density

Definition. In a poset $\mathbb{P}$, we say that a subset F is a **filter** iff it satisfies the following two properties:

- If $p, q \in F$, then there exists $r \in F$ with $r \leq p, q$.
- For all $p, q \in \mathbb{P}$, if $p \leq q$ and $p \in F$, then $q \in F$.

In posets, we will often call p a "strengthening" of q whenever we have $p \leq q$; in this sense, the first condition is the statement that says that p and q have a "common strengthening." In this situation, we will call p and q "compatible:" this should feel intuitive in the poset that we're working with, as two functions are in this sense "compatible" iff they can be combined together into some function that is an extension of both of them.

As well, we say that sets that satisfy the second of these condtions are "closed upwards:" filters in this language, then, are collections of closed-upwards functions that are all compatible.

Our collection G, in specific, is a great example of a filter.

This takes care of the first two "nice" properties of G: what about the others? We address them here:

Definition. Given a poset $\mathbb{P}$ and a subset D of $\mathbb{P}$, we say that D is **dense** iff for any $p \in \mathbb{P}$, there exists a strengthening $q \leq p$ such that $q \in D$.

Notice that the two sets

$$D_n := \{(\varphi, \mathscr{F}_0) : n \in \mathrm{dom}(\varphi)\}$$

and

$$D_f := \{(\varphi, \mathscr{F}_0) : f \in \mathscr{F}_0\}$$

are both dense in our poset $\mathbb{P}_{\mathscr{F}}$. Why is this? Well, take any pair (φ, F_0) in $\mathbb{P}_{\mathscr{F}}$. To get something stronger than it that's in D_n, just extend it on all of the values $\leq n$ that it's not defined on in such a way that we keep all of the promises made in $\mathscr{F}_0$. There are finitely many n and finitely many promises, so this is trivial: therefore, this extended function φ' can be combined with our original set of promises $\mathscr{F}_0$ such that $(\varphi, \mathscr{F}_0) \geq (\varphi', \mathscr{F}_0)$ and $(\varphi', \mathscr{F}_0)$ is in D_n.

Similarly, to get something that's a strengthening of $(\varphi, \mathscr{F}_0)$ in D_f, just take $(\varphi, \mathscr{F}_0 \cup \{f\})$: this is trivially a strengthening of our original element, and also something in D_f. So these sets are dense!

With this language defined, we can rephrase our earlier observations about G as follows:

- G is a filter.
- G has nontrivial intersection with the dense sets D_n, D_f, for every $n \in \omega$, $f \in \mathscr{F}$.

So: given this, a somewhat natural question to ask is the following: is this even possible? I.e. if I give you a collection of dense sets, can you find a filter that intersects them all?

In other words, consider the following possible statement:

Axiom 3 *If $\mathbb{P}$ is a poset and $\{D_\alpha \mid \alpha < \kappa < |2^\omega|\}$ is a collection of $< |2^\omega|$ dense sets, then there exists a filter $G \subseteq \mathbb{P}$ such that $G \cap D_\alpha \neq \emptyset$ for all $\alpha < \kappa$.*

Is this true in ZFC? Is this even consistent with ZFC? Could this be the axiom we're looking for?

No!

Come back tomorrow, and we'll tell you why this is full of lies.