Martin's Axiom	Instructors: Susan Durst and Padraic Bartlett

Lecture 2: Martin's Axiom, For Reals This Time

Week 5 — *Mathcamp 2011*

We did a lot of things last lecture! We quickly recap them here:

- We defined the concept of a **dominating** function for a set of functions $\mathscr{F} \subset^{\omega} \omega$, and asked whether these things always existed for sets $\mathscr{F}$ with cardinality strictly between ω and 2^{ω}.
- To study this, we introduced a poset $\mathbb{P}_{\mathscr{F}} = \{(\varphi, \mathscr{F}_0)\}$, where the φ's are all functions $\{1, \ldots n_{\phi}\} \to \omega$ and the sets $\mathscr{F}_0 \subset \mathscr{F}$ were all finite sets of "promises" that our function ϕ would keep in the future. To make this idea concrete in our poset, we said that $(\varphi, \mathscr{F}_0) \geq (\psi, \mathscr{F}_1)$ iff the following held:
  - ψ was an extension of φ: i.e. $\mathrm{dom}(\varphi) \subseteq \mathrm{dom}(\psi)$, and they agree wherever they're both defined.
  - $\mathscr{F}_0 \subset \mathscr{F}_1$.
  - ψ, thought of as an extension of φ, keeps all of φ's promises on its newly-defined values: i.e. $\psi(m) > f(m)$, for any $m \in (\mathrm{dom}(\psi) \setminus \mathrm{dom}(\varphi))$, $f \in \mathscr{F}_0$.
- To get a better idea of how to turn this poset into a dominating function, we assumed that we had some such dominating function and looked at what it told us about the poset. Specifically, we looked at the set G of all pairs $(\varphi, \mathscr{F}_0)$ such that φ was a finite piece of g and g satisfied all of $\mathscr{F}_0$'s promises on values not yet defined by φ. This set had two remarkable properties:
  - It was a **filter**[1].
  - It had nontrivial intersection with all of the **dense**[2] sets $D_f = \{(\varphi, \mathscr{F}_0) : f \in \mathscr{F}_0\}$ and $D_n = \{(\varphi, \mathscr{F}_0) : n \in \mathrm{dom}(\varphi)\}$
- Motivated by this, we suggested the following form of $MA(\kappa)$, Martin's axiom for a given cardinal κ: If $\mathbb{P}$ is a poset and $\{D_\alpha \mid \alpha < \kappa < |2^{\omega}|\}$ is a collection of $< |2^{\omega}|$ dense sets, then there exists a filter $G \subseteq \mathbb{P}$ such that $G \cap D_\alpha \neq \emptyset$ for all $\alpha < \kappa$.
- We then said two things about this axiom:
  1. This axiom is awesome, because we can turn filters into functions!
  2. This axiom is false.

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[1] In a poset $\mathbb{P}$, we say that a subset F is a **filter** iff it satisfies the following two properties:

∗ If $p, q \in F$, then there exists $r \in F$ with $r \leq p, q$.

∗ For all $p, q \in \mathbb{P}$, if $p \leq q$ and $p \in F$, then $q \in F$.

[2] Given a poset $\mathbb{P}$ and a subset D of $\mathbb{P}$, we say that D is **dense** iff for any $p \in \mathbb{P}$, there exists a strengthening $q \leq p$ such that $q \in D$.

To open up our lecture today, we're going to prove both of these things: that we can turn filters into functions, and this axiom is blatantly false. For fun, let's do them at the same time!

1 The Axiom Is A Lie

Specifically, consider the following poset $\mathbb{P}$ and dense sets D_α, D_n:

- Elements of $\mathbb{P}$ are objects of the form ϕ, where ϕ is a function from $\{1, \ldots n_\phi\}$ to ω_1, the first uncountable ordinal. The ordering on $\mathbb{P}$ is $\varphi \geq \psi$ iff ψ is an extension of φ.
- For any $\alpha \in \omega_1$, define D_α as the collection of all of the elements ϕ that have α in their image.
- For any $n \in \omega$, define D_n as the collection of all of the elements ϕ that have n in their domain.

Suppose that our axiom above holds for ω_1! Then there is some filter G that intersects all of these dense sets. We want to do two things with it:

- Use it to build a function.
- Show that the function it built is **full of fail**.

Let's do the first here. Given this filter G, pick for every n a pair $\varphi_n \in G$ such that φ_n has n in its domain: we know that these exist because $G \cap D_n \neq \emptyset$, for every n. Having done this, define

$$g(n) = \varphi_n(n).$$

Now, notice a few things about this function g:

1. First, did it matter which function we picked when we chose our φ_n's? No! This is because if we take any two functions φ_n, φ_n' with n in their domain, we know that they have to have a common strengthening ψ, and therefore they had both better agree at n!
2. Second, for any $\alpha \in \omega_1$, pick some element φ_α in our filter that has α in its range. (This exists because $D_\alpha \cap G$ is nonempty!) Let n be the value such that $\varphi_\alpha(n) = \alpha$. Then, by our first observation, we know that $g(n) = \alpha$, because our choice of φ_n didn't matter!

So. What did we do? We made a function (yay!) from ω to ω_1 that, um, is surjective (boo!) In other words, we demonstrated (1) that filters make functions, and (2) that our proposed axiom is way too powerful to exist in its current form.

2 Fixing It!

We still like our axiom, though! It certainly gives us a function that seems promising; however, the issue with the above construction was that there were, in some sense "too many poorly-behaved" functions in our poset. In other words, we had tons of things in our poset that couldn't be compared and were doing wildly different things (like hitting different elements of ω_1,) and we were somehow trying to assemble a function out of all of them that would combine them together.

To fix this, we want to introduce a condition on our poset that stops this "too many poorly-behaved" function phenomena from happening: in other words, we want to have our poset have a bound on how many incompatible objects live in it. We do this by introducing the following property:

Definition. In a poset $\mathbb{P}$, we say that two elements p, q are **compatible** iff they have some common refinement $r \leq p, q$. Otherwise, we say that p and q are incompatible.

A poset $\mathbb{P}$ is said to have the **countable chain condition**, abbreviated CCC, if and only if every antichain[3] in $\mathbb{P}$ is countable.

To illustrate just what the countable chain condition is after, we study some sample posets:

Proposition 1 *The partial order $\mathbb{P}_{\mathscr{F}}$ we've been studying throughout this class satisfies the CCC.*

Proof. We study this in two parts.

First, recall that the functions φ are all maps from some finite set $\{0, \ldots n_\varphi\}$ to ω. How many such functions exist? Well: for any n, there are only countably many maps $\{0, \ldots n\} \to \omega$. Therefore, if we take the union over all n, we have a countable union of countable things, which is countable! So there are only countably many such φ.

But what does this mean? Well, take any antichain A in $\mathbb{P}_{\mathscr{F}}$. Because any two elements $(\varphi, \mathscr{F}_0)$ and $(\varphi, \mathscr{F}_1)$ have a common refinement $(\varphi, \mathscr{F}_0 \cup \mathscr{F}_1)$, we know that any two elements $(\varphi, \mathscr{F}_0), (\psi, \mathscr{F}_1)$ in A must have $\varphi \neq \psi$. But this means that we have at most countably many elements in A, as there are only countably many such functions.

Proposition 2 *The partial order $\mathbb{P}$ on functions $\omega \to \omega_1$ we defined earlier in this lecture does not satisfy the CCC.*

Proof. Just take

$$A = \bigcup_{\alpha \in \omega_1} \{\varphi_\alpha : \mathrm{dom}(\varphi_\alpha) = \{0\}, \varphi_\alpha(0) = \alpha\}.$$

So: The CCC doesn't upset the example we're working on, but clearly disallows the awful counterexample we found to our first draft of our axiom. Excellent! Let's throw it in our axiom:

[3] An antichain in a poset is a collection of elements that are all pairwise incompatible. Yes, the countable chain condition is a statement about antichains. Yes, it certainly **should** have been called the countable antichain condition. No, I have no idea why it is not.

Axiom 3 *(MA(κ), Martin's Axiom for a cardinal κ:) If $\mathbb{P}$ is a nonempty poset with the CCC and $\mathscr{D} = \{D_\alpha\}$ is a collection of $\leq \kappa$-many dense sets, then there exists a filter $G \subseteq \mathbb{P}$ such that $G \cap D_\alpha \neq \emptyset$ for all $D \in \mathscr{D}$.*

3 Fixed It!

To kind-of illustrate what's going on $MA(\kappa)$, let's try plugging in some values of κ and seeing what happens:

Theorem 4 *$MA(\omega)$ is true.*

Proof. Let $\mathbb{P}$ be our nonempty poset with CCC, and $\{D_i\}_{i=1}^{\infty}$ our collection of dense sets. In order from each D_i, using the density of the D_i's, pick an element p_i such that

$$p_0 \geq p_1 \geq p_2 \geq \dots$$

Let

$$G = \{x \in \mathbb{P} : \exists\ n \text{ s.t. } x \geq p_n\}.$$

We claim that this is a filter. Closed upwards is trivial, by its definition: so it suffices to show that any two elements in it have a common refinement. Take any two elements $x, y \in G$, and let p_x, p_y be the elements are respectively refinements of them. Then we have that one of p_x, p_y is a refinement of the other, and thus (by transitivity) that x and y have a common refinement in G.

(As an aside: even though this class is centered around applying MA(κ) for values of $\kappa > \omega$, in practice Martin's axiom is frequently used in the form $MA(\omega)$, as it allows you to do some fairly crazy things with countable models of ZFC! Talk to us in TAU if you're curious.)

Theorem 5 *$MA(2^\omega)$ is false.*

Proof. To see this, let $\mathscr{F} =^\omega \omega$, and consider our poset $\mathbb{P}_\mathscr{F}$ that we've been working with thus far. For every $f \in^\omega \omega$, let D_f be the dense set consisting of all of the pairs $(\varphi, \mathscr{F}_0)$ where $f \in \mathscr{F}_0$. As well, for every n let D_n be the dense set containing all of the pairs $(\varphi, \mathscr{F}_0)$, where $n \in \text{dom}(\varphi)$.

Then, if there is a filter G that intersects all of these D_f's, we can just do the same trick we did in constructing our "bad" function earlier in class! Specifically:

- Using the dense sets D_n, find pairs $(\varphi_n, \mathscr{F}_{0,n})$ for every n such that $n \in \text{dom}(\varphi_n)$, and define a function $g(n) = \varphi_n(n)$.
- As noted before, these choices of $\varphi_n(n)$ didn't really matter: any two possible choices of $\varphi_n(n)$ are the same, because they have a common refinement!
- To see why this might be an issue, for every $f \in^\omega \omega$ pick $(\varphi_f, \mathscr{F}_{0,f})$ in our filter. For any $n \notin \text{dom}(\varphi_f)$, we know that $(\varphi_f, \mathscr{F}_{0,f})$ and $(\varphi_n, \mathscr{F}_{0,n})$ have a common refinement $(\psi, \mathscr{F}_1)$. By definition, we know that $\psi(n) > f(n)$, and therefore that $g(n) > f(n)$ because our choices of $\varphi_n(n)$'s didn't matter!

So we've created a function $g \in^\omega \omega$ that dominates ... every function in ${}^\omega\omega$. Including itself. Fail!

Theorem 6 *$MA(\kappa)$, for $|\omega| < \kappa < |2^\omega|$ is independent of the ZFC axioms.*

(We don't have space for a proof of this here, but interested students should find us during TAU!)

Ok! That's definitely some pleasantly axiom-like behavior; much better than our first draft, which was just false. So: because it's so beautiful, let's assume that $MA(\kappa)$ holds. Given this, can we answer our dominating function question?

Theorem 7 *Assume $MA(\kappa)$ holds. Take any subset $\mathscr{F} \subseteq^\omega \omega$. Then there is a dominating function for $\mathscr{F}$.*

Proof. Basically, we're going to perform the same tricks we used in our two disproofs earlier to construct functions that were awful. But this time, it's gonna work!

More formally:

- Again, use the dense sets D_n to find pairs $(\varphi_n, \mathscr{F}_{0,n})$ for every n such that $n \in \text{dom}(\varphi_n)$. Define $g(n) = \varphi_n(n)$.
- As noted twice before, these choices of $\varphi_n(n)$ didn't really matter: any two possible choices of $\varphi_n(n)$ are the same, because they have a common refinement!
- We want to show that this function g dominates every element of $\mathscr{F}$. To do this, take any $f \in \mathscr{F}$, and (using the density of D_f) find a pair $(\varphi_f, \mathscr{F}_{0,f})$ in our filter such that $f \in F_{0,f}$.
- For any $n \notin \text{dom}(\varphi_f)$, we know that $(\varphi_f, \mathscr{F}_{0,f})$ and $(\varphi_n, \mathscr{F}_{0,n})$ have a common refinement $(\psi, \mathscr{F}_1)$. By definition, we know that $\psi(n) > f(n)$, and therefore that $g(n) > f(n)$ because our choices of $\varphi_n(n)$'s didn't matter! So g dominates f.

This time, we've just created a function g that dominates every element of $\mathscr{F}$! Win!

Excellent! So, in a certain very-specialized sense, we've just shown that assuming Martin's axiom makes cardinalities of sets between ω and 2^ω act "countable-ish:" in other words, we've shown that whenever Martin's axiom holds, we have that this property of countable sets (being able to find dominating functions) is one that holds for κ-sets as well!

Perhaps surprisingly, this phenomena is not simply restricted to dominating functions! You can use Martin's axiom to prove that κ-cardinalities are like ω in lots of ways, including the following:

- Assume MA(κ). Then, the union of κ-many sets of measure 0 is a set of measure 0.
- Assume MA(κ). Then $|2^\kappa| = |2^\omega|$.

Beautiful, right?