Perfect Graph Theory	Instructor: Padraic Bartlett

Lecture 2: An Introduction to Perfect Graphs

Week 1	*Mathcamp 2011*

Last lecture, in our attempt to "classify" the graphs of high chromatic number in a similar way to how we classified all of the bipartite graphs, we saw that having a high chromatic number can be a purely global phenomenon: via the Mycielski construction, we were able to construct graphs with arbitrarily high chromatic number that locally looked like trees (i.e. that locally looked 2-colorable.)

As a result, classifying **all** of these graphs seems like it will be an incredibly difficult task. Any such classification of graphs of high chromatic number will have to capture both the graphs that have high $\chi(G)$ because they contain a large clique (i.e. graphs where $\chi(G)$ is large because of local phenomena) and also graphs where $\chi(G)$ is large because of some global phenomena, which we at the moment barely understand at all / lack a good characterization of.

As mathematicians, then, what should we do? Well: one thing that we often do in mathematics, when we're trying to classify a set of objects, several of which are pathological or otherwise troublesome, is to simply **ignore the pathological things**! I.e.:

- In calculus/analysis, we often restrict our attention to "continuous functions," or "differentiable functions," rather than work with the entire space of all functions. This is because certain operations like integration or differentiation can only be done on sufficiently nice objects, and (rather than weaken the notions of integration and differentiation) it was much more fruitful to simply restrict the class of functions that we study.

- In algebra, when we move from studying groups to abelian groups, or abelian groups to rings, or rings to fields, we're ostensibly restricting the space of objects that we're looking at. In exchange, however, we can suddenly can do a lot more with these objects every time we "add" constraints: while we are studying a smaller class of objects each time, we are also able to say much more every time we add some additional constraints (as they remove certain "pathologies," like noncommutative rings, from the things we have to consider.)

How can we do this for our problem?

1 Defining Perfect Graphs

As stated above, the issue we ran into when trying to classify graphs based on their chromatic number was that $\chi(G)$ could become a purely global phenomena: i.e. $\chi(G)$ could be far far larger than $\omega(G)$, the clique number of G. Motivated by this, and our above discussion, we define the family of **perfect graphs** to be the graphs where this does not happen:

Definition. A graph G is called **perfect** if and only if $\chi(H) = \omega(H)$, for every induced subgraph H of G.

Perfect graphs, then, are graphs where the chromatic number is a purely local phenomenon: i.e. to show that $\chi(G) \geq k$, we can simply find some K_k sitting inside of G. (This is in sharp contrast to the situation with general graphs, where we'd have to go through every single possible $k-1$ coloring and prove by contradiction that none of those colorings could be proper.) This class of graphs, then, avoids the pathologies of the Mycielski graphs and other examples; as a result, we might be able to classify these graphs in a similar fashion to the bipartite graphs!

First, we should try to show that such graphs **exist**:

2 Several Examples

The most trivial class of graphs that are perfect are the **edgeless graphs**, i.e. the graphs with $V = \{1, \dots n\}$ and $E = \emptyset$; these graphs and all of their subgraphs have both chromatic number and clique number 1.

Only slightly less trivially, we have that the **complete graphs** K_n are all perfect. This is because any induced subgraph H of K_n on k vertices is itself a complete graph on k vertices; therefore, we have that $k = \chi(H) = \omega(H)$, for any such H.

The last class of graphs we can trivially notice are perfect are the **bipartite graphs** $G = ((V_1, V_2), E)$. This is because of the following two observations:

1. if G is not edgeless, then we have $\chi(G) = 2 = \omega(G)$, because bipartite graphs don't contain any odd cycles (and therefore, in specific, no K_3's.)

2. If H is an induced subgraph of a bipartite graph G, it is either edgeless or bipartite.

Therefore, we have that $\chi(H) = \omega(H)$ for every subgraph H of a bipartite graph, and thus that it must be bipartite.

More interestingly, we have the following proposition:

Proposition 1 *If G is a bipartite graph, then the complement graph*[1] *$\overline{G}$ of G is perfect.*

Proof. Let $G = ((V_1, V_2), E)$ be a bipartite graph on n vertices, $\overline{G}$ be its complement graph, and $\overline{H}$ be any induced subgraph of $\overline{G}$ on the vertex set $\{v_1^1, \dots v_k^1\} \cup \{v_1^2, \dots v_l^2\}$. Then, $\overline{H}$ is the graph with edges between all of the v_i^1's in it, as well as edges between all of the v_j^2's in it, and also an edge between a v_i^1 and a v_j^2 iff such an edge did not exist in G.

Consequently, by definition, $\overline{H}$ is itself the complement graph of the induced subgraph H of G given by the same vertex set $\{v_1^1, \dots v_k^1\} \cup \{v_1^2, \dots v_l^2\}$, which is a bipartite graph. Therefore, any subgraph of $\overline{H}$ is itself complement graph of a bipartite graph

Thus, if we can prove that $\chi(\overline{G}) = \omega(\overline{G})$ for any complement graph to a bipartite graph, then we will have proven that this in fact holds for all of G's subgraphs (as they are all also complement graphs to bipartite graphs) and thus shown that $\overline{G}$ is perfect.

[1] The complement graph $\overline{G}$ of a graph G is the graph consisting of all of the edges that weren't in G: i.e. $V(G) = V(\overline{G})$, and $e \in E(G)$ iff $e \notin E(\overline{G})$

To do this, simply note that in any coloring of $\overline{G}$, every color class consists of either one single vertex or two vertices that were adjacent in G (because G does not contain any triangles, there are no sets of three vertices in $\overline{G}$ that **don't** have at least one edge between them.) Therefore, $\chi(\overline{G})$ is the smallest number of combined edges and vertices in G it takes to cover all of G without any overlaps (i.e. your color classes are the edges and vertices in this coloring.)

What is this quantity? As it turns out, this is answered by the following proposition, which is interesting in its own right:

Proposition 2 *If G is a bipartite graph with no isolated vertices, then $\alpha(G)$, the size of the largest independent set*[2] *of vertices in G, is equal to $\beta'(G)$, the size of the smallest edge covering*[3] *of G.*

Proof. On the HW!

For right now, let's focus on the application of this theorem. First, notice that $\alpha(G) = \omega(\overline{G})$, as the largest clique in the complement graph corresponds to the largest set of independent vertices in the original graph. Then, because $\alpha(G) = \beta'(G)$, it thus suffices to show that we can turn a minimal edge covering of G into a covering of G with edges and vertices, so that no vertex is contained in two things.

But this is pretty easy. Let E be any minimal edge covering of G. Take any two edges in E that overlap on a given vertex, and replace one of those edges by its other endpoint: this produces an edge and a vertex that do not overlap. Keep doing this until no edges overlap; at the end of this process, you will have a collection of vertices and edges with no overlap, of the same size as $\beta'(G)$.

As mentioned before, this forms a proper coloring of $\overline{G}$; so we've just colored $\overline{G}$ with $\beta'(G) = \alpha(G) = \omega(\overline{G})$-many colors! This tells us that $\chi(\overline{G}) \leq \omega(\overline{G})$; however, we trivially have $\omega(\overline{G}) \leq \chi(\overline{G})$ for any graph, so we can conclude

$$\chi(\overline{G}) = \omega(\overline{G}),$$

and thus that $\overline{G}$ is perfect, as claimed.

We close by discussing one last example of perfect graphs for today:

Definition. If G is a graph, we can form the **line graph** $L(G) = H$ from G as follows:

- The vertex set of H is the edge set $E(G)$ of G.
- There is an edge between two elements of $V(H)$ if and only if the two corresponding edges in $E(G)$ share a common vertex.

In other words, it's the graph formed by taking all of the edges in G and treating them as vertices, and connecting these edge-vertices whenever any two of them shared a vertex in G.

[2] A set of vertices S is called **independent** in a graph G iff there are no edges between elements of S in G.

[3] A set of edges T is called a **covering** of a graph G iff every vertex of G lies in at least one edge of T.

Proposition 3 *If G is a bipartite graph, then $L(G)$ is perfect.*

Proof. Notice that (just like in the last two examples) any induced subgraph of $L(G)$ is also a line graph of some bipartite graph, as any induced subgraph of $L(G)$ corresponds to the line graph on some arbitrary subgraph of G, which is still bipartite. Therefore, it again suffices to prove that $\chi(L(G)) = \omega(L(G))$.

How can we calculate $\chi(L(G))$? Well: what we're really trying to do here is break G's edges up into different color classes, so that no two incident edges are the same color. One way to do this is to recall Hall's marriage theorem, which said that

Theorem 4 *(Hall) If $G = ((V_1, V_2), E)$ is a bipartite graph, then G has a matching*[4] *of V_1 into V_2 if and only if $|N(S)| \geq |S|$, for any subset $S \subset V_1$.*

One easy application of this is the following:

Proposition 5 *If $G = ((V_1, V_2), E)$ is a bipartite graph, we can divide G's edges up into $\Delta(G)$ many sets $A_1, \ldots A_{\Delta(G)}$, such that all of the edges in any A_i are disjoint.*

Proof. First, add vertices to either V_1 or V_2 so that $|V_1| = |V_2|$. Then, by connecting pairs of vertices $v_1 \in V_1, v_2 \in V_2$ where $\deg(v_1), \deg(v_2) < \Delta(G)$, add edges to G so that it is $\Delta(G)$-regular[5].

We now claim that for any subset S of V_1, we have $|N(S)| \geq |S|$. To see this, notice that because there are $|S| \cdot \Delta(G)$-many edges leaving S and entering $N(S)$, there must be some vertex in $N(S)$ receiving at least $|S| \cdot \Delta(G)/|N(S)|$-many edges. Because all of the degrees of vertices in G are equal to $\Delta(G)$, this forces $|N(S)| \geq |S|$.

Therefore, by Hall's theorem, there is a matching of V_1 and V_2; because these are the same size, this is a set of $|V_1|$ edges, all disjoint, covering all of G's vertices. Take this collection, call it $A_{\Delta(G)}$, and delete these edges from G. We now have a $(\Delta(G) - 1)$-regular graph.

Repeat this process $\Delta(G)$ many times; it will create $\Delta(G)$-many sets A_i, such that their union contains all of the edges of G and all of the edges in any A_i are disjoint. By discarding the extra edges and vertices we threw in at the start of our construction, we have our claimed partition of G's edges.

This proves that $\chi(L(G)) \leq \Delta(G)$, as the above division gives us a $\Delta(G)$-coloring. As well, we know that $\omega(L(G)) \geq \Delta(G)$; this is because if we look at any vertex with degree $\Delta(G)$ in G, it corresponds to a $K_{\Delta(G)}$ in the line graph of G (as all $\Delta(G)$ of these edges are connected to each other.) Again, because we trivially know that $\omega(L(G)) \leq \chi(L(G))$, we've proven that $\chi(L(G)) = \omega(L(G))$, and thus that $L(G)$ is perfect.

[4] For a bipartite graph $G = ((V_1, V_2), E)$, a **matching** of V_1 into V_2 is a set of edges in G such that each element of V_1 is used exactly once, and no two edges share any endpoints in common.

[5] A graph G is called k-regular if and only if every vertex in G has degree k.