Perfect Graph Theory	Instructor: Padraic Bartlett

Lecture 4: The Perfect Graph Theorem, Part 1/2

Week 1	*Mathcamp 2011*

One property you may have noticed in day 2's lecture was that everything that we characterized as being perfect – complete graphs, bipartite graphs, the line graphs of bipartite graphs – also had perfect complements! In the next two lectures, we will prove that this is always the case; in other words, we will prove that a graph G is perfect if and only if its complement $\overline{G}$ is.

1 The Perfect Graph Theorem: Preliminaries

The result that a graph is perfect if and only if its complement is perfect is called the **Perfect Graph Theorem**. There are several proofs of it in existence; we will focus on the way it was proven originally, by Lovász and Fulkerson in 1970.

First, we need to prove some preliminary results, which are interesting in their own right:

Proposition 1 *A graph is perfect if and only if every induced subgraph H has an independent set that intersects every clique in H of maximal order (i.e. order $\omega(H)$.) In other words, for any subgraph H, there's an independent set I such that*

$$\omega(H - I) < \omega(H).$$

Proof. As this is an if and only if statement, we must prove both directions.

Let G be a perfect graph. Because any induced subgraph of G is perfect, it suffices to find I such that $\omega(G - I) < \omega(G)$. Doing this is trivial, however: just take any $\chi(G) = \omega(G)$ coloring of G, and let I be one of the color classes used in this coloring. This set is independent by definition; as well, because we've removed one color, $G - I$ satisfies $\omega(G - I) = \chi(G - I) < \chi(G) = \omega(G)$. Thus, we have proven this direction of our claim.

Now, take a graph G such that every induced subgraph $H \subset G$ has an independent set I_H intersecting every clique in H of maximal order. We seek to show that such a graph is perfect, and do so by inducting on $\omega(G)$. As the only graphs with $\omega(G) = 1$ are the edgeless graphs, there is nothing to prove here; so we assume that $\omega(G) = n$ and that we've proven our result for all values of $n' < n$.

Let H be any induced subgraph of G, and let I be an independent set of vertices in H such that $\omega(H - I) < \omega(H)$. By our inductive hypothesis, $H - I$ is perfect, and therefore $\chi(H - I) = \omega(H - I)$; so we can color $H - I$ with $\omega(H - I)$-many colors. Take any such coloring, and extend it to a coloring of H by coloring I some other, new color; this gives us a coloring of H with $\omega(H - I) + 1$ many colors. Because $\omega(H - I) < \omega(H)$, this means that our coloring of H uses $\leq \omega(H)$ many colors: i.e. that $\omega(H) \leq \chi(H)$. But this means that $\omega(H) = \chi(H)$.

As this holds for every subgraph H of G, we've proven that G is perfect, as claimed.

Using this alternate characterization of perfect graphs, we can prove the following result, which (in addition to being useful in our proof of the Perfect Graph Theorem) gives us a remarkably useful way to create new graphs:

Theorem 2 *A graph obtained from a perfect graph by replacing any of its vertices with a perfect graph is still perfect.*

Proof. First, we should say what we mean by "replacing a vertex with a graph:"

Definition. Given a graph G_1, a vertex $\alpha \in V(G_1)$, and another graph G_2, we can define the graph G^* created by **substituting** G_2 **for** α as the following:

- Vertex set of $G*$: $V(G_1) \setminus \{\alpha\}$, unioned with $V(G_2)$.
- Edge set of G^*: $\{u, w\} \in E(G^*)$ if and only if either $\{u, w\} \in E(G_1 \setminus \{\alpha\}), \{u, w\} \in E(G_2)$, or $u \in V(G_1 \setminus \{\alpha\}), w \in V(G_2)$, and there was an edge from u to α in $E(G_1)$.

Basically, this is the graph formed by taking G_1 and replacing α with an entirely new graph G_2, with edges drawn to all of G_2 whenever there were edges involving α.

Notice that this process can only increase the clique and chromatic number of the graph G_1.

Let G_1, G_2 be a pair of perfect graphs, α be a vertex in G_1, and G^* be the graph formed by replacing α with G_2. Consider any induced subgraph H of G_1, with vertex set $V_1 \cup V_2$, $V_1 \subset V(G_1), V_2 \subseteq V(G_2)$. Because both of the induced subgraphs on the sets V_1 and V_2 are perfect, by definition, we can see that H is also a graph formed by taking a perfect graph and replacing a vertex with another perfect graph.

Using our earlier proposition, then, it suffices to prove that for any such graph G^*, we can find an independent set I of $\omega(G^*)$ vertices that interesects every maximal clique of G^*. To construct this set, simply do the following:

- Take a $\omega(G_1)$ coloring of G_1, and let J be the color class of G_1 that contains our vertex α. Remove the element α from J.
- Using our earlier proposition, find an independent set K in the graph G_2 that non-trivially intersects every clique in G_2 of order $\omega(G_2)$.
- Let $I = J \cup K$. Notice that this set is clearly independent in G^*: there are no edges between elements of J or elements of K by definition, as both are independent sets; as well, there are no edges between elements of J and K, as α was colored the same color as the elements in J, and we replaced α with G_2 to make G^*.

Let L be any clique in G^* with maximal size $\omega(G^*)$; it now suffices to show that L and I intersect nontrivially. There are two cases:

1. L is contained entirely in $G_1 \setminus \{\alpha\}$. If this happens, then (because $\omega(G^*) \geq \omega(G_1) = \chi(G_1)$), L must contain at least one vertex from every color class in $G_1 \setminus \{\alpha\}$. This is because L is a clique, and therefore cannot contain two elements of the same color under any proper coloring. This then means that L and J share an element in common, and thus that L intersects I nontrivially.

2. There are some vertices of L that lie within G_2. In this case, because L is a clique of maximal size, this portion of L that lies in G_2 should be of maximal size for a clique in G_2: in other words, $|L \cap V(G_2)| = \omega(G_2)$. By definition, then, we know that K must intersect this clique, and thus that $I = J \cup K$ must also intersect this clique, which is part of L.

Therefore, we've shown that any maximal clique intersects our independent set; thus, by our earlier proposition, G^* is perfect.