Lecture 5: The Perfect Graph Theorem, Part 2/2

Last lecture, we presented a pair of results to help us prove the perfect graph theorem. We restate them here:

Proposition 1 *A graph is perfect if and only if every induced subgraph H has an independent set that intersects every clique in H of maximal order (i.e. order $\omega(H)$.) In other words, for any subgraph H, there's an independent set I such that*

$$\omega(H - I) < \omega(H).$$

Theorem 2 *A graph obtained from a perfect graph by replacing any of its vertices with a perfect graph is still perfect.*

With this machinery, we are now ready to prove the perfect graph theorem:

1 The Perfect Graph Theorem

Theorem 3 *(Lovász) A graph G is perfect if and only if its complement is perfect.*

Proof. We proceed by induction on the number of vertices in G; again, as always, the case $n = 1$ is trivial, and we can proceed to the inductive step where we take $|V(G)| = n$. By our earlier proposition, in order to show that $\overline{G}$ is perfect, we just need to show that $\overline{G}$ contains an independent set I that intersects every clique in $\overline{G}$ whenever G is perfect (this suffices because every induced subgraph of G is perfect, and therefore every induced subgraph of $\overline{G}$ is the complement of a perfect graph.)

Because working with the complement graph is kind of awkward, let's rephrase this condition in terms of properties of G:

- If I is an independent set in $\overline{G}$, then, in G, it corresponds to a clique.
- If I intersects every maximal clique in $\overline{G}$, then in G it intersects every maximal independent set of vertices: i.e. if J is an independent set of vertices such that $|J| = \alpha(G)$, then I intersects J nontrivially.

We proceed by contradiction: suppose that no such set exists. Then, for every complete subgraph K contained within G, there must be some independent set J such that K and J do not intersect. Let $L_1, \ldots L_r$ be all of the possible subgraphs of G isomorphic to complete graphs and let $I_1, \ldots I_r$ denote their corresponding independent sets.

What can we do with this? Well, we are eventually attempting to arrive at a contradiction, which (given that our central assumption is $\omega(G) = \chi(G)$) will probably look something like $\chi(G) > \omega(G)$. So: what can we do with our graph to show that its chromatic number is "too large?"

One bound you may remember from earlier graph theory courses, that looks plausibly relevant, is the following:

$$\chi(G) \geq \frac{|V(G)|}{\alpha(G)}.$$

(This is because any k-coloring of G's vertices induces a division of G into k independent sets; because none of them can be greater than $\alpha(G)$, we have the bound indicated above.)

At the moment, this isn't too relevant, as we don't have any good relation between $|V(G)|$ and $\omega(G)$. However, we can create one! Specifically: let's use our perfect graph substitution theorem, that we had earlier!

This theorem allows us to take vertices in G and replace them with perfect graphs. For ease of calculations, we probably want to make sure this graph has the same independence number as G: what kinds of graphs will insure this? Well, in general, the only graphs you can add in that won't possibly increase the independence number are complete graphs (which we've shown to be perfect;) so let's try adding those in! Specifically: let's replace every vertex $v_1, \ldots v_n$ in G with a complete graph K_{v_i} on $i(v_j)$-many vertices (where we'll decide what these values $i(v_j)$ are later,) and let's call the resulting graph G^*.

What can we now say about $\omega(G^*)$? Well, any complete subgraph of G^* is obtained by first taking a complete subgraph of G, and then adding at most $i(v)$-many vertices for each vertex in the complete subgraph. So, if we look at our list $L_1, \ldots L_t$ of our complete subgraphs of G and pick one (say L_r) with maximal order $= \omega(G)$, we then have that

$$\omega(G^*) = \sum_{v \in L_r} i(v)$$

As well, if we now turn and look at the chromatic number, by applying our earlier bound (plus the observation that $\alpha(G^*) = \alpha(G)$, we also have

$$\chi(G^*) \geq \frac{\sum_{j=1}^{n} i(v_j)}{\alpha(G)}.$$

How can we relate these sums, the first of which relates to the elements in a complete subgraph and the second of which relates to the independence number of our graph, via a clever choice of our values $i(v_i)$? After some clever thinking, one promising idea comes up: set $i(v_j)$ to be the number of independent sets I_k containing v_j! One immediate reason to like this is because it nicely simplifies our sum above: indeed, because $\sum_{j=1}^{n} i(v_j) = \sum_{j=1}^{t} |I_j| = t \cdot \alpha(G)$, we have that in fact

$$\chi(G^*) \geq \frac{\sum_{j=1}^{n} i(v_j)}{\alpha(G)} = t.$$

What does this mean for our bound on $\omega(G^*)$? Well, we can rewrite

$$\omega(G^*) = \sum_{v \in L_r} i(v) = \sum_{v \in L_r} \left(\sum_{j : v \in I_j} 1 \right) = \sum_{j=1}^{t} |L_r \cap I_j|.$$

But L_r intersects each I_j at most once (because L_r is a clique,) and never intersects I_r at all: therefore, we have that this sum is at most the number of I_j's minus 1; i.e. $t-1$.

So, we've shown that $\omega(G^*) \leq t-1$, and that $\chi(G^*) \geq t$. But G^* is perfect, because we arrived at it by replacing vertices with complete graphs! This is a contradiction.

Therefore, we've proven that G has a clique that meets every maximal independent set; i.e. that $\overline{G}$ has an independent sets that meets every maximal clique, which means that (via our proposition and earlier discussion) that $\overline{G}$ is perfect.

2 Further Results and Directions

There are a number of other results whose proofs we won't quite have time for in this course. One result is slightly suggested by the proof above: there, we showed that

$$\omega(G) = \chi(G) \geq \frac{|V(G)|}{\alpha(G)}.$$

As it turns out, this trivially necessary condition is actually sufficient for a graph to be perfect: in other words, we have the following theorem:

Theorem 4 *(Gasparian) A graph G is perfect if and only if for every induced subgraph H of G, we have*

$$\omega(H) = \chi(H) \geq \frac{|V(H)|}{\alpha(H)}.$$

The proof of this implies the perfect graph theorem (as its statement is symmetric in G and $\overline{G}$, something that's more easily seen by writing $\alpha(H) = \omega(\overline{H})$, and multiplying through to the expression $|H| = \omega(H)\omega(\overline{H})$.)

Another result, this one by Lovász, was motivated by the idea of seeing what we can get by "relaxing" the idea of independence. In other words, one way to define an independent set is the following:

Definition. A **independent set of vertices** can be thought of as a function $f : V(G) \to \{0, 1\}$ such that

$$\sum_{v \in K} f(v) \leq 1,$$

for any complete subgraph K of G. (The actual set in question is the collection of values on which f is 1, here.)

Note that we can now define $\alpha(G)$ as the maximum of f over all of $V(G)$: i.e.

$$\alpha(G) = \max_f \sum_{v \in V(G)} f(v)$$

Now, if we relax our definition of f to let it take on **any** values between 0 and 1, we can define a **fractionally independent set**:

Definition. A **fractionally independent set of vertices** can be thought of as a function $f : V(G)$ to the interval $[0, 1]$, such that

$$\sum_{v \in K} f(v) \leq 1,$$

for any complete subgraph K of G. (There is no longer really a well-defined "set" here: just a collection of weights we've assigned to all of the elements in $V(G)$.

Under this definition, we can define the **fractional independence number** of G, $\alpha^*(G)$, to be the maximum value of such functions of f on G: i.e.

$$\alpha^*(G) = \max_f \sum_{v \in V(G)} f(v).$$

Lovasz's theorem is the following surprising result:

Theorem 5 *A graph is perfect if and only if $\alpha^*(H) = \alpha(H)$, for every induced subgraph H.*

Finally, we return to our original question motivating this entire study: Is there a way to classify perfect graphs based on subgraphs they don't have (like we did with bipartite graphs?) Because induced subgraphs of perfect graphs are perfect, it would suffice to actually just classify the family of **critically imperfect graphs**: graphs that aren't perfect, but all of whose induced subgraphs are perfect.

All of the odd cycles of length ≥ 5, and by extension their complements, are critically imperfect; surprisingly, it's rather hard to find other examples. So surprising, in fact, that Berge (one of the pioneers of perfect graph theory) created the following conjecture:

Theorem 6 *A graph G is perfect iff neither G nor its complement contain an induced odd cycle of length ≥ 5.*

You may have noticed that I wrote "theorem" here, instead of "conjecture;" this is because this was proven in 2002, by Chudnovsky, Robertson, Seymour, Thomas, a few months before Berge passed away. Its proof is incredibly long (150+ pages), and is mostly structural (i.e. they decompose perfect graphs into many different cases, seeking to classify them as all having certain kinds of special "structures" that can be shown to not contain induced odd cycles of length ≥ 5.)