Perfect Graph Theory — Instructor: Padraic Bartlett

Homework 1

Week 1 — *Mathcamp 2011*

Attempt the problems that seem interesting! Easier exercises are marked with $(-)$ signs; harder ones are marked by $(*)$. Open questions are denoted by writing $(**)$, as they are presumably quite hard.

1. $(-)$ Show that if a graph G has chromatic number k, then it must have at least $\binom{k}{2}$ edges.

2. If a graph on n vertices has chromatic number $\leq r$, what's the most edges it can have? Is there a unique graph with this many edges? (Hint: consider the complete r-partite graphs, where each part has size $\sim r/n$.)

3. Let G be a k-chromatic graph with girth ≥ 6, with vertex set $\{v_1, \dots v_n\}$. Construct a new graph G' as follows:

   - Let T be a set of kn vertices, $\{t_1, \dots t_{kn}\}$ with no edges between them.
   - Take $\binom{kn}{n}$ disjoint copies of G, one for every n-subset of $\{1, \dots kn\}$ and index them by these subsets: i.e. for any subset $\{i_1, \dots i_n\} \subseteq \{1, \dots kn\}$, make a subgraph $G_{\{i_1, \dots i_n\}}$.
   - Take each $G_{\{i_1, \dots i_n\}}$, and connect the vertices of G to the corresponding vertices in T given by G's indexing subset. In other words, throw in the edges $\{v_1, t_{i_1}\}, \{v_2, t_{i_2}\}, \dots \{v_n, t_{i_n}\}$ to our graph made by the the G's and the set T.

   Show that this graph still has girth 6, as well as chromatic number ≥ 6.

4. $(-)$ Start with a P_2 and draw the next graph created by the above process.