Perfect Graph Theory	Instructor: Padraic Bartlett
Homework 2	
Week 1	*Mathcamp 2011*

The problems below are completely optional; attempt the ones that seem interesting to you! Easier exercises are marked with $(-)$ signs; harder ones are marked by $(*)$. Open questions are denoted by writing $(**)$, as they are presumably quite hard.

1. Let G be a graph such that $\chi(G \setminus \{x, y\}) = \chi(G) - 2$, for all vertices in G. Show that G must be the complete graph.

2. $(**)$ Suppose that G is a graph such that $\chi(G \setminus \{x, y\}) = \chi(G) - 2$, for all pairs of **adjacent** vertices in G. Show that G must be the complete graph. (This has been resolved for $k \leq 5$, and is open for $k = 6$ and higher, though some results are known! These graphs are called **double-critical graphs**, as an aside.)

3. A **partially ordered set** $P = (X, <)$ is a collection of vertices $\{x_1, \ldots x_n\}$ that satisfies the following two properties:

   - **Antisymmetry**: if $x < y$, we do not have $y < x$.
   - **Transitivity**: if $x < y$ and $y < z$, we have $x < z$.

   Given a partially ordered set $P = (X, <)$, we can construct the **comparability graph** G_P corresponding to this set, by having $V(G_P) = X$, and $E(G_P) = \{\{x, y\} : x < y$ or $y < x\}$. Show that every compararbility graph is perfect.

4. In a partially ordered set $P = (X, <)$, a **chain** is a sequence of elements $x_1 < \ldots < x_n$; conversely, an **antichain** is a set $S \subset X$ such that no two elements in S are comparable (i.e. if $x < y$, then either x or y (or both!) are not in S.) Prove **Dilworth's theorem**: if every antichain in P has less than m elements, then P can be written as the union of m chains.

5. (-) Use the above theorem to prove that the complement graphs of comparability graphs are also perfect.