Perfect Graph Theory | Instructor: Padraic Bartlett

Homework 4

Week 1 | *Mathcamp 2011*

The problems below are completely optional; attempt the ones that seem interesting to you! Easier exercises are marked with $(-)$ signs; harder ones are marked by $(*)$. Open questions are denoted by writing $(**)$, as they are presumably quite hard.

1. $(-)$ A graph G is called **minimally imperfect** if $\omega(G) \neq \chi(G)$, but for any induced subgraph H of G, $\omega(H) = \chi(H)$. Show that if a graph is imperfect, that it must contain a minimally imperfect subgraph.

2. $(-)$ Show that any odd cycle of length ≥ 5 is minimally imperfect.

3. $(*)$ A **cutset** S of a graph G is a collection of vertices S such that $G \setminus S$ is disconnected. A **star-cutset** S of a graph G is a cutset such that there is some vertex $x \in S$, such that x has an edge to every other element in S.

   Using these definitions, prove the **Star-Cutset Lemma Lemma**: Suppose you have any graph G such that (1) every proper induced subgraph of G is $\omega(G)$-colorable, and (2) G does not contain an indepedent set that intersects all of its maximal cliques. Then G does not contain a star-cutset.

4. $(-)$ Using the Star-Cutset Lemma Lemma, prove the **Star-Cutset Lemma**: No minimally imperfect graph has a star-cutset.

5. Use the Star-Cutset Lemma to prove that taking a perfect graph and replacing one of its vertices with another perfect graph is perfect.