Perfect Graph Theory	Instructor: Padraic Bartlett
Homework 5	
Week 1	*Mathcamp 2011*

The problems below are completely optional; attempt the ones that seem interesting to you! Easier exercises are marked with $(-)$ signs; harder ones are marked by $(*)$. Open questions are denoted by writing $(**)$, as they are presumably quite hard.

1. $(*)$ Prove Gasparian's theorem: A graph G is perfect if and only if for every induced subgraph H of G, we have

$$\chi(H) \geq \frac{|V(H)|}{\alpha(H)}.$$

Hint: Proceed by induction on $|V(H)|$. By induction, you know that every induced subgraph of G is perfect; so it suffices to prove that $\chi(G) = \omega(G)$, for G a graph on the vertex set $\{1, \dots n\}$. Now, proceed by contradiction: assume that $\chi(G) > \omega(G)$.

Prove that if this is true, then if U is any independent set in G, we have $\chi(G \setminus U) = \omega(G \setminus U) = \omega(G)$.

Having done that let $A_0 = \{u_1, \dots u_\alpha\}$ be an independent set in G of size $\alpha = \alpha(G)$; for each u_i, let $A_{1+i\omega}, \dots A_{(i+1_\omega}$ be the ω-different color classes of a proper ω coloring of $G \setminus \{u_i\}$. As well, for each A_i there is a complete graph on ω vertices in $G \setminus A_i$: call this graph K_i.

Show that $K_i \cap A_j$ is empty for exactly one j.

From here (and this is the twist!), let J be the $\alpha\omega + 1$ by $\alpha\omega + 1$ matrix with 0's down its diagonal and 1's everywhere else; let A be the real $\alpha\omega + 1 \times n$ matrix whose rows are the incidence vectors of the A_i's with elements in $V(G)$; and let B be the real $n \times \alpha\omega + 1$ matrix whose rows are the incidence vectors of the K_i's with elements in $V(G)$.

Prove that

$$J = AB.$$

Conclude that

$$\chi(G)\omega(G) + 1 \leq |V(H)|,$$

and thus that there is a contradiction.

2. Use the above theorem to prove that if G is a minimally imperfect graph on n vertices, then

$$\chi(G)\alpha(G) + 1 = n.$$

3. Prove any of the old problems! Alternately, if you've already done them all/want something tricky to think about, consider the following construction I mentioned on Monday:

1 Bonus: Nešetřil and Rödl's Construction

We state the construction here, and leave it to the reader to supply the proof:

Definition. A k-**hypergraph** $G = (V, \mathcal{E})$ consists of the following:

- V, a collection of vertices, and
- $\mathcal{E}$, a collection of subsets of V of size k, all of which are distinct.

Basically, this is a generalization of the idea of a graph, where we're saying that an edge consists of k elements, and not just 2. Simple graphs are just 2-hypergraphs, for instance.

Definition. A **cycle** in a k-hypergraph $G = (V, \mathcal{E})$ is a collection of edges $M_1, \dots M_n$ such that $M_i \cap M_{i+1 \mod n}$ is nonempty, for every $1 \leq i \leq n$.

Definition. A k-hypergraph $G = (V, \mathcal{E})$ is said to be a-**partite** or a-**colorable** if and only if there is a way to color the vertices of G with a colors, $\{1, 2, \dots a\}$, such that no edge $E \in \mathcal{E}$ contains two or more vertices of the same color. We say that $\chi(G) = a$ if a is the smallest value such that G is a-colorable.

Definition. Let $G = (V, \mathcal{E})$ be a k-hypergraph with $\chi(G) \leq a$, f be a proper a-coloring of G, r be a fixed color from the set $\{1, \dots a\}$, and K be the number of vertices colored r in G. Let $H = (W, \mathcal{F})$ be a K-hypergraph.

Then, define the r-**amalgamation** $(W, \mathcal{F}) * (V, \mathcal{E})$ of these two hypergraphs as the following a-colorable k-hypergraph $(X, \mathcal{Y})$:

- Let V_i denote all of the vertices in V colored i under our coloring map f.
- Let $X_r = W$, and $X_i = V_i \times \mathcal{F}$, where this product is understood as the Cartesian set product.
- Let X, our vertex set, be the disjoint union of these X_i's, and let g be the coloring of these vertices given by our subscripts.
- For every edge $F \in \mathcal{F}$, pick a bijection $\iota_F : X_r \to F$. It doesn't matter what you pick here, so long as we fix one for every edge.
- Finally, we say that a k subset Y of X is an edge in $\mathcal{Y}$ if and only if the following holds: There are a pair of edges $E \in \mathcal{E}$, $F \in \mathcal{F}$ such that
  - $Y' \cap X_r = \iota_F(E \cap V_r)$, and
  - $Y \cap X_i = (E \cap V_i, F)$.

This sounds kind of awful, but in reality all we're doing is taking $|\mathcal{F}|$ many identical copies of $(V, \mathcal{E})$, and identifying the copies of X_r with $(W, \mathcal{F})$.

With these definitions out of the way, we can finally proceed to our theorem:

Theorem 1 *There are k-hypergraphs with chromatic number $\geq n$ and girth $\geq p$, for any k, n, p.*

Proof. Fix any value of n and k: we prove our statement by inducting on p. For $p = 1$ our statement is trivially true, as cycles of length 1 do not exist; now, assume that we've proven our claim for all $p' \leq p$. We seek to construct a k-hypergraph with chromatic number $\geq n$ and no cycles of length $p + 1$ or smaller.

Let $a = (k-1)n + 1$. We inductively create a family of $a + 1$ different a-colorable k-hypergraphs, as follows:

- Let $(V^0, \mathcal{E})$ be the a-colored hypergraph defined by
  - $V^0 = \{1 \dots a\}^k \times \{1, \dots k\}$
  - $V_i^0 = \{(c_1, \dots c_k, j) : c_j = i\}$
  - $\mathcal{E} = \{\{(c_1, \dots c_k, 1), (c_1, \dots c_k, 2), \dots (c_1, \dots c_k, k)\}$: the c_i's are k distinct fixed colors. $\}$.

  The upshot of this is that this graph is a-colorable, has no paths in it at all, and yet for any choice of k distinct colors has an edge with all of those colors in it.
- Given a graph $(\{V_j^i\}_{j=1}^{n+1}, \mathcal{E})$, construct a new graph as follows:
  - Let $|V_i^i| = K_i$.
  - Using our inductive hypothesis, let $(W^i, \mathcal{F}^i)$ be a K_i-hypergraph without any cycles of length $\leq p$.
  - Define $(\{V_j^{i+1}\}_{j=1}^{n+1}, \mathcal{E})$ as the amalgamation $(W^i, \mathcal{F}^i) * (V^i, \mathcal{E})$.

Running this process a times yields a graph $(\{V_j^{a+1}\}_{j=1}^{n+1}, \mathcal{E})$. You can show that this graph has chromatic number $\geq n$ and girth $\geq p + 1$ without much more difficulty!