Spectral Graph Theory	Instructor: Padraic Bartlett

Lecture 3: Turning Eigenvalues Into Information

Week 3 | *Mathcamp 2011*

Last lecture/in TAU, I had a number of students ask me just what these eigenvalues "represent " in a graph. Today, we'll start answering that question!

1 Basic Applications of Eigenvalues

In our introduction to graph theory course, pretty much the first interesting thing we did was characterize bipartite graphs. As it turns out, we can do the same thing here in terms of the spectra:

Proposition 1 *If a graph G is bipartite, its spectrum is symmetric about 0; that is, for any bipartite graph G, λ is an eigenvalue of A_G if and only if $-\lambda$ is.*

Proof. Write $G = (V_1 \cup V_2, E)$,, where $V_1 = \{1, 2, \ldots k\}$ and $V_2 = \{k+1, k+2, \ldots n\}$ partition G's vertices. In this form, we know that the only edges in our graph are from V_1 to V_2. Consequently, this means that A_G is of the form

$$\begin{array}{|c|c|}\hline 0 & B \\ \hline B^T & 0 \\ \hline \end{array},$$

where the upper-left hand 0 is a $k \times k$ matrix, the lower-right hand 0 is a $n-k \times n-k$ matrix, B is a $(n-(k+1)) \times k$ matrix, and B^T is the transpose of this matrix.

Choose any eigenvalue λ and any eigenvector $(v_1, \ldots v_k, w_{k+1}, \ldots w_n) = (\mathbf{v}, \mathbf{w})$. Then, we have

$$A_G \cdot (\mathbf{v}, \mathbf{w}) = \begin{array}{|c|c|}\hline 0 & B \\ \hline B^T & 0 \\ \hline \end{array} \cdot \begin{bmatrix} \mathbf{v} \\ \mathbf{w} \end{bmatrix} = \begin{bmatrix} B \cdot \mathbf{w} \\ B^T \cdot \mathbf{v} \end{bmatrix} = \begin{bmatrix} \lambda \cdot \mathbf{v} \\ \lambda \cdot \mathbf{w} \end{bmatrix} = \lambda \begin{bmatrix} \mathbf{v} \\ \mathbf{w} \end{bmatrix}$$

But! This is not the only eigenvector we can make out of $\mathbf{v}$ and $\mathbf{w}$. Specifically, notice that if we multiply A_G by the vector $(\mathbf{v}, -\mathbf{w})$, we get

$$A_G \cdot (\mathbf{v}, -\mathbf{w}) = \begin{array}{|c|c|}\hline 0 & B \\ \hline B^T & 0 \\ \hline \end{array} \cdot \begin{bmatrix} \mathbf{v} \\ -\mathbf{w} \end{bmatrix} = \begin{bmatrix} -B \cdot \mathbf{w} \\ B^T \cdot \mathbf{v} \end{bmatrix} = \begin{bmatrix} -\lambda \cdot \mathbf{v} \\ \lambda \cdot \mathbf{w} \end{bmatrix} = -\lambda \begin{bmatrix} \mathbf{v} \\ -\mathbf{w} \end{bmatrix}.$$

In other words, whenever λ is an eigenvalue of A_G, $-\lambda$ is as well!

As well, one of the next things we studied was how certain key properties (like $\chi(G)$) depended on the maximum degree of the graph, $\Delta(G)$. As this provides an upper bound on the overall density of our graph, it seems like a natural candidate for something to bound our eigenvalues by. We make this explicit in the following proposition:

Proposition 2 *If G is a graph and λ is an eigenvalue of A_G, then $|\lambda| \leq \Delta(G)$.*

Proof. Take any eigenvalue λ of A_G, and let $\mathbf{v} = (v_1, \dots v_n)$ be a corresponding eigenvector to λ. Let v_k be the largest coördinate in $\mathbf{v}$, and (by scaling $\mathbf{v}$ if necessary) insure that $v_k = 1$.

We seek to show that $|\lambda| \leq \Delta(G)$. To see this, simply look at the quantity $|\lambda \cdot v_k|$. On one hand, we trivially have that this is $|\lambda|$.

On the other, we can use the observation that $\mathbf{v}$ is an eigenvector to notice that

$$\begin{aligned}
|\lambda \cdot v_k| &= |(a_{k1}, a_{k2}, \dots a_{kn}) \cdot (v_1, \dots v_n)| \\
&= \left|\sum_{j=1}^{n} a_{kj} v_j\right| \leq \left|\sum_{j=1}^{n} a_{kj} v_k\right| = \left|v_k \sum_{j=1}^{n} a_{kj}\right| \\
&\leq |v_k \cdot \Delta(G)| = \Delta(G).
\end{aligned}$$

When is the above bound tight? With many graph properties (like, again, $\chi(G)$,) answering this question is usually difficult. Here, however, it's actually quite doable!, as we demonstrate in the next proposition:

Proposition 3 *A connected graph G is regular if and only if $\Delta(G)$ is an eigenvalue of A_G.*

Proof. As this is an if and only if, we have two directions to prove.

($\Rightarrow$:) If G is regular, then each vertex in G has degree $\Delta(G)$. This means (amongst other things) that there are precisely $\Delta(G)$ 1's in every row of A_G. Consequently, if we look at $A_G \cdot (1, 1, 1 \dots 1)$, we know that we get the vector $(\Delta(G), \Delta(G), \dots \Delta(G))$: i.e. $\Delta(G)$ is an eigenvector!

($\Leftarrow$:) As before, pick an eigenvector $\mathbf{v}$ for our eigenvalue $\Delta(G)$, let v_k be the largest component of $\mathbf{v}$, and rescale $\mathbf{v}$ so that $v_k = 1$. Then, just as before, we have

$$\begin{aligned}
|\Delta(G)| &= |\Delta(G) \cdot v_k| = |(a_{k1}, \dots a_{kn}) \cdot (v_1, \dots v_n)| \\
&= \left|\sum_{i=1}^{n} a_{ki} v_i\right| \leq \left|\sum_{i=1}^{n} a_{ki} v_k\right| = |v_k| \left|\sum_{i=1}^{n} a_{ki}\right| = \deg(v_k) \\
&\leq \Delta(G),
\end{aligned}$$

and therefore $\deg(v) = \Delta(G)$.

But wait! If the above is true, we actually have

$$|\Delta(G)| = \left|\sum_{i=1}^{n} a_{ki} v_i\right| = \Delta(G),$$

and therefore v_i is equal to 1 for **every** i adjacent to k! Therefore, we can repeat the above argument for every i adjacent to k, and show that the degree of all of these vertices are also $\Delta(G)$. Repeating this process multiple times shows that the degree of **every** vertex is $\Delta(G)$, and therefore that G is regular.

Proposition 4 *If G is a graph with diameter*[1] $d \in \mathbb{N}$, *then A_G has at least $d+1$ distinct eigenvalues.*

Proof. First, recall the spectral theorem, which says that (because A_G is real-valued and symmetric) we can find an orthonormal basis for $\mathbb{R}^n$ made out of A_G's eigenvectors. Let $\vec{e}_1, \ldots \vec{e}_n$ be such a basis of orthonormal eigenvectors, and let the collection of distinct eigenvalues of A_G be $\theta_1, \ldots \theta_t$.

Examine the product $D = (A_G - \theta_1 \cdot I) \cdot (A_G - \theta_2 \cdot I) \cdot \ldots (A_G - \theta_t \cdot I)$; specifically, notice that the order of the θ_i's doesn't matter in this above product, as

$$\begin{aligned}&(A_G - \theta_1 \cdot I) \cdot (A_G - \theta_2 \cdot I) \cdot \ldots (A_G - \theta_t \cdot I)\\ =&A_G^t - \left(\sum_{i=1}^{n} \theta_i\right) A_G^{t-1} + \ldots + (-1)^t \prod_{i=1}^{t} \theta_i \cdot I,\end{aligned}$$

and the θs in each of the coefficients above clearly commute.

What happens when we multiply D on the right by any of these $\vec{e}_{i_t}$'s? Well: if we permute the $(A_G - \theta I)$'s around so that $(A_G - \theta_{i_t} I)$ is the first term, we have

$$\begin{aligned}&(A_G - \theta_{i_1} \cdot I) \cdot (A_G - \theta_{i_2} \cdot I) \cdot \ldots (A_G - \theta_{i_t} \cdot I) \cdot e_{i_t}\\ =&(A_G - \theta_{i_1} \cdot I) \cdot (A_G - \theta_{i_2} \cdot I) \cdot \ldots \cdot (A_G \cdot e_{i_t} - (\theta_{i_t} \cdot I) \cdot e_{i_t})\\ =&(A_G - \theta_{i_1} \cdot I) \cdot (A_G - \theta_{i_2} \cdot I) \cdot \ldots \cdot 0\\ =&0.\end{aligned}$$

But this means that D sends **all** of the $\vec{e}_i$'s to 0: i.e. that D sends all of these basis vectors for $\mathbb{R}^n$ to 0! In other words, this forces $D = 0$.

But what does this mean? Using our expansion above of D into a polynomial, we've just shown that

$$A_G^t - \left(\sum_{i=1}^{n} \theta_i\right) A_G^{t-1} + \ldots + \prod_{i=1}^{t} \theta_i \cdot I = 0,$$

which means that

$$A_G^t = \left(\sum_{i=1}^{n} \theta_i\right) A_G^{t-1} + \ldots - \prod_{i=1}^{t} \theta_i \cdot I.$$

What would happen if the diameter d is greater than $t-1$? It would mean that there are two vertices i, j such that $d(i, j) = t$, at the very least! But this would mean that the

[1] The **diameter** of a graph G is the longest distance between any two vertices in a graph.

(i,j)-th entry of A_G^t would be nonzero (as there is a path of length t between them), while the (i,j)-th entry of A_G^k would be zero for every $k < t$ (as there are no paths of shorter length linking them.) But this is impossible, because we've written A_G^t as the sum of these smaller matrices!

Therefore, this cannot occur: i.e. we have $d \leq t - 1$, which is what we wanted to prove.