Spectral Graph Theory	Instructor: Padraic Bartlett
Lecture 5: The Matrix-Tree Theorem	
Week 3	*Mathcamp 2011*

This lecture is also going to be awesome, but shorter, because we're finishing up yesterday's proof with the first half of lecture today.

So: a result we've proven in like 3-4 MC classes this year, in different ways, is the following:

Theorem 1 *(Cayley) There are n^{n-2} labeled trees on n vertices.*

Today, we're going to prove the ridiculously tricked-out version of this theorem:

Theorem 2 *(The Matrix-Tree Theorem) Suppose we have any graph G. Let L_G denote the Laplacian[1] of G. Let Γ denote the number of spanning trees[2] of G: then, we have*

$$\Gamma = \frac{1}{n} \cdot \mu_2 \cdot \ldots \mu_n,$$

where $\mu_1 \leq \mu_2 \leq \ldots \leq \mu_n$ are the n eigenvalues of the Laplacian of G written in increasing order, and we've removed the first one of these from our product above.

Proof. Before we begin, we first review some key facts about the Laplacian:

1. The Laplacian is real-valued and symmetric: so it has n eigenvalues counting multiplicity, by the spectral theorem.

2. The Laplacian has 0 as an eigenvalue: this is because summing any of its rows yields 0, and therefore the all-1's vector is an eigenvalue for 0.

3. The Laplacian is positive-definite[3], and therefore all of its eigenvalues are ≥ 0.

Prove these things on the HW, if you don't believe them!

Also: for notational clarity, let $L^{\{v_1,\ldots v_k\}}$ denote the matrix L_G if we delete the k rows and columns corresponding to these vectors, and $l_{x,x}$ denote the quantity $\det(L^{\{x\}})$. We now proceed to prove our claim in two parts: first, we claim that

$$\Gamma = l_{x,x}$$

for any $x \in V(G)$.

We prove this claim by a pair of nested inductions: first on the number n of vertices in G, and then (at each level) by inducting on the number of edges in this n-vertex graph.

The first case where our notation makes sense is $n = 2$: there, we have that $l_{x,x}$ is simply the degree of the other remaining vertex, which is either 0 (in which case our graph

[1] The Laplacian of a graph G is the $n \times n$ matrix with rows/columns indexed by vertices, with a -1 in every (i, j) where an edge runs from (i, j), the degree of vertex i in the entry (i, i), and 0's elsewhere.

[2] A **spanning tree** of a graph G is a subgraph that uses every vertex in G and is also a tree.

[3] A matrix A is called **positive-semidefinite** iff $\mathbf{x}^T(A\mathbf{x}) \geq 0$, for any vector $\mathbf{x}$.

is disconnected and no spanning trees exist) or 1 (in which case this one edge forms the unique spanning tree.) So our claim holds here.

We assume that we've proven our case for all $k < n$, and proceed to n vertices. In the case where there are no edges leaving the vertex x, we are trivially done: $\Gamma = 0$ because $\{x\}$ is disconnected from the graph, while $L^{\{x\}}$ is just L_G where we've removed an all-zero row and column, which is therefore a matrix that still has zero row sums (and thus one whose determimant, $l_{x,x}$, is 0.)

Otherwise, there is an edge involving x: denote it as $\{x, y\}$. How does deleting this edge from L_G change $l_{x,x}$? Well: deleting this edge decreases the (x, x) and (y, y)-th entries of L_G by 1, and increases the (x, y) and (y, x) entries of L_G to 0. However, in $L^{\{x\}}$, the only one of those changes that we can still see is the decrement of the (y, y)-th entry by 1, as we deleted the row and column involving x!

Expanding, if we denote this modified matrix as M, we can see that

$$\begin{aligned}
\det(M) &= (-1)^{y-1} \det(M, \text{with row } y \text{ switched to the top}) \\
&= (-1)^{y-1} \sum_{1 \le i \le n-1} (-1)^{i-1} \cdot m_{i,y} \cdot \det(M, \text{row } y \text{ at top, row } y \text{ and col } j \text{ deleted}) \\
&= (-1)^{y-1} \sum_{1 \le i \le n-1} (-1)^{i} \cdot m_{i,y} \cdot \det(M \text{ with row } y \text{ and col } j \text{ deleted}) \\
&= -(-1)^{2y-2} \cdot \det M_{y,y} + (-1)^{y-1} \sum_{1 \le i \le n-1, i \ne y} (-1)^{i} \cdot m_{i,y} \cdot \det(M_{y,j}) \\
&= -\det(L^{\{x,y\}}) + l_{x,x}.
\end{aligned}$$

Thus, we've shown that removing an edge from G decreases $l_{x,x}$ by $\det(L^{\{x,y\}})$, which (by induction) is the number of spanning trees on the graph G if we contracted the edge $\{x, y\}$ to a point. But this is just the number of spanning trees on G that specifically use the edge $\{x, y\}$. Therefore, by repeatedly doing this process, our inductive claim holds (i.e. we've proven that $\Gamma = l_{x,x}$.)

To finish this proof, we just need to do the following two things, which you will prove on the HW:

1. Notice that because we can factor the characteristic polynomial $\det(xI - L)$ by its roots, we have that
$$\frac{\partial}{\partial x} (\det(xI - L)) \Big|_{x=0} = (-1)^{n-1} \cdot \mu_2 \cdot \ldots \cdot \mu_n.$$
2. Conversely: notice as well that
$$\frac{\partial}{\partial x} (\det(xI - L)) = \sum_{x=1}^{n} \det(tI - L^{\{x\}}),$$
and thus that when we plug in zero to the above equation, we get $(-1)^{n-1} \cdot \sum_{x=1}^{n} l_{x,x}$.

Combining these two observations with our earlier one that $\Gamma = l_{x,x}$ gives us that

$$\Gamma = \frac{1}{n} \cdot \mu_2 \cdot \ldots \mu_n,$$

as claimed.