Latin Squares — Instructor: Padraic Bartlett

Homework 2: Partial Latin Squares, continued

Week 2 — *Mathcamp 2012*

Attempt all of the problems that seem interesting, and let me know if you see any typos! (+) problems are harder than the others. (++) problems are currently open.

1. Is the following partial Latin square P always completeable to a proper Latin square? (Assume that P is of order ≥ 3.)

$$\begin{bmatrix} 1 & - & - & \dots & - \\ - & 2 & - & \dots & - \\ - & - & 3 & \dots & - \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ - & - & - & \dots & n \end{bmatrix}$$

2. Another way to interpret a (partial) Latin square L as a graph is the following construction:

   - Start with a set of $3n$ vertices. Label n of them $r_1, \dots r_n$, and think of these elements as corresponding to the rows of L; take n more different vertices, label them $c_1 \dots c_n$, and think of them as corresponding to the columns, and take the last n, label them $s_1 \dots s_n$, and think of these as corresponding to the symbols.
   - Every time the symbol k occurs in cell (i, j), draw a triangle connecting r_i, c_j, and s_k.
   - This creates a **tripartite**[1] graph. Furthermore, it does this in a way that subdivides our graph up into a bunch of edge-disjoint triangle subgraphs!

   Take the following partial Latin square P and turn it into a tripartite graph:

$$P = \begin{bmatrix} - & 2 & 3 \\ 2 & 1 & - \\ 3 & - & 1 \end{bmatrix}$$

   Draw the tripartite complement of this graph: i.e. the tripartite graph formed by connecting r_i to s_j, or s_j to c_k, or c_k to r_i, if and only if we did not connect them with an edge in our earlier construction. What kind of graph is this? How is this related to the question of completing P to a Latin square? (I.e. just by looking at this graph, can you explain why P cannot be completed to a proper Latin square?)

3. Yesterday, I asked you for the smallest number of cells that you could place in a 4×4 partial Latin square, so that it has a unique solution. Today, do the opposite: find the largest number of cells you can place in a 4×4 partial Latin square, so that it cannot be completed to a proper Latin square. Does your construction generalize to $n \times n$ Latin squares?

[1] A tripartite graph is one in which the vertex set can be split into three parts V_1, V_2, V_3, such that there are no edges that start and end in the same V_i. Like bipartite, but with three parts!

4. (+) Let P be a partial Latin square that satisfies the following property: there is a set of r rows and c columns such that a cell in P is filled if and only if it lies within the intersection of these rows and columns. Then P is completeable if and only if $N(i) \geq r + c - n$, where $N(i)$ denotes the number of symbols in P equal to i.

5. An slightly easier version of the above question: show that if P is a $n \times n$ partial Latin square, n even, where the upper-quadrant $\frac{n}{2} \times \frac{n}{2}$ is filled and the rest is blank, then P can be completed to a Latin square.