Latin Squares Instructor: Padraic Bartlett

Lecture 6: Latin Squares and the n-Queens Problem

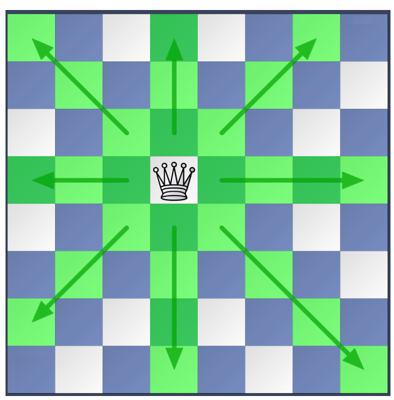
Week 3 *Mathcamp 2012*

In our last lecture, we introduced the idea of a **diagonal Latin square** to help us study magic squares. As it turns out, this idea of a "diagonal" Latin square is useful for more than just magic squares; we can use a similar and stronger concept to study the n-queens problem, a classical question from chess and mathematics that we discuss in this lecture.

1 The n-Queens Problem

If you haven't played chess, here's a quick summary of the things you will need to understand for this lecture:

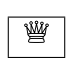
- **A** $n \times n$ **chessboard** is simply a $n \times n$ array of cells.
- A **queen** in the game of chess is a piece, shaped like ♕. In the game of chess, when moved, a queen (when placed in a given cell in a chessboard) can go to any cell within the same row, any cell within the same column, or any cell along the two diagonals through the cell that it starts from. We illustrate this here:



Given this terminology, we can now state the n-queens problem, which is the following:

Question. Take a $n \times n$ chessboard. Can you place n distinct queens on this chessboard, so that no queen can capture any other (i.e. so that there is no way to move any one queen into a cell currently occupied by another queen?)

Again, like last time, we can decide whether or not this problem is doable for small values of n by just trying to do it for small values. For $n = 1$, this is pretty trivial to do: behold!



For $n = 2$, it's also pretty easy to see that this is impossible. Any one queen in a 2×2 board can move to any other square; therefore, it is impossible to place a second queen into our grid.

♕	×
×	×

For $n = 3$, it's also not too hard to see that this is impossible. When you place a ♕ on a 3×3 board, it either goes on in the center (in which case it can move to any other square) or one of the side/corner squares, in which case there are precisely two squares to which it cannot move. Any second queen placed in either of those spaces can move to the other space; therefore, we cannot place a third queen.

♕	×	×
×	×	
×		×

$\to$

♕	×	×
×	×	×
×	♕	×

,

×	×	
♕	×	×
×	×	

$\to$

×	×	×
♕	×	×
×	×	♕

,

×	×	×
×	♕	×
×	×	×

.

For $n = 4$, however, we can do it! Consider the following arrangement:

×	×	♕	×
♕	×	×	×
×	×	×	♕
×	♕	×	×

The above pattern you can find by just exhaustively searching through possible ways to place queens on a 4×4 chessboard. In general, however, we want a systematic approach; i.e. a pattern that we can follow to always solve the n-queens problem, or tell us that no such solution exists.

Surprisingly enough, it turns out that we already have a solution! Well, at least, for infinitely many values of n. Consider the following definition:

Definition. A **broken right diagonal**, or **wraparound right diagonal**, in a Latin square L is the set of n cells acquired by starting from one of the cells in our top row and repeatedly taking the cell that's one below and one to the right of this cell, wrapping around our square if we hit the last column, until we get to the last row.

		×		
			×	
				×
×				
	×			

A **broken left diagonal** is the same kind of object, except wrapping around to the left instead of the right.

Given these definitions, a Latin square L is called **pandiagonal** (alternately, **diabolic**, or **perfect**, depending on the author) if every broken diagonal contains no repeated symbols.

Pandiagonal Latin squares should feel like a much stronger version of the diagonal Latin squares we studied earlier; as opposed to just requiring that the main diagonal and antidiagonal contain no repeats, we now require **every** diagonal, including the broken ones, to not have any repeats.

The reason we care about these is the following observation:

Proposition. Suppose that L is a $n \times n$ pandiagonal Latin square. Choose any symbol s occuring in L. Take a $n \times n$ chessboard, and place a queen on every cell containing s. Then this is a solution to the n-queens problem.

Proof. Because L is pandiagonal, there are no repeated symbols s in any row, column, or broken diagonal: therefore, in particular, if we place a queen at every cell containing a symbol s, none of these queens can move to a cell containing another queen.

So: we care about pandiagonal Latin squares! The next natural question we could ask is whether these even exist; but, as it turns out, we already constructed a ton of these yesterday! Recall the following construction:

Construction. Take any value of n, and any two numbers $a, b \in \{0, \ldots n-1\}$. Consider the following square populated with the elements $\{0, 1 \ldots n-1\}$:

$$L = \begin{array}{|c|c|c|c|c|c|} \hline 0 & a & 2a & 3a & \ldots & (n-1)a \\ \hline b & b+a & b+2a & b+3a & \ldots & b+(n-1)a \\ \hline 2b & 2b+a & 2(b+a) & 2b+3a & \ldots & 2b+(n-1)a \\ \hline 3b & 3b+a & 3b+2a & 3(b+a) & \ldots & 3b+(n-1)a \\ \hline \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \hline (n-1)b & (n-1)b+a & (n-1)b+2a & (n-1)b+3a & \ldots & (n-1)(b+a) \\ \hline \end{array} \mod n.$$

In other words, L's (i, j)-th cell contains the symbol given by taking the quantity $ai + bj$ mod n.

Yesterday, we showed that this was a diagonal Latin square whenever $a, b, a+b, a-b$ were relatively prime to n. However, we actually have much more! Consider any broken right diagonal. If we're careful with how we write it mod n, we can see that it's actually of the form

	$\ldots$	ka		$\ldots$	
	$\ldots$		$ka + (a+b)$	$\ldots$	
	$\vdots$			$\ddots$	
	$\ldots$			$\ldots$	$ka + (n-k-1)(a+b)$
$ka + (n-k)(a+b) \ldots$					
	$\ddots$				
	$\ldots$	$ka + (n-1)(a+b)$			

Therefore, all of these entries are distinct if $a + b$ is relatively prime to n. Similarly, all of the entries in any broken left diagonal are also distinct if $a - b$ is relatively prime to n;

therefore, our construction from before actually creates not just diagonal Latin squares, but pandiagonal Latin squares! We can use such a grid to solve the 5-queens problem:

0	2	4	1	3
1	3	0	2	4
2	4	1	3	0
3	0	2	4	1
4	1	3	0	2

$\longrightarrow$

♕	×	×	×	×
×	×	♕	×	×
×	×	×	×	♕
×	♕	×	×	×
×	×	×	♕	×

However, this construction only worked when n was neither even nor a multiple of 3. Consequently, we initially would want to try to extend our construction to these cases; either by trying to modify our construction to somehow work on even values of n, or just by trying new constructions that maybe would become pandiagonal Latin squares.

After a while of this, we'd probably get frustrated. As mathematicians, the most fun thing to do when you're frustrated with proving something is to turn around and prove that it's impossible! Let's do that:

Claim 1 *No pandiagonal Latin squares exist of even order.*

We prove this claim via the following observations and definitions:

Definition. A **transversal** in a Latin square is a way to pick out n cells so that every row, column, and symbol is represented by exactly one cell. An example is highlighted below:

1	**3**	4	2
4	2	1	3
2	4	3	**1**
3	1	**2**	4

Lemma 2 *If a Latin square L has an orthogonal mate, its cells can be broken up into n disjoint transversals.*

Proof. Take L and its orthogonal mate M. Given any symbol k occuring in M, look at all of the cells containing k in M. These cells in L cannot contain any repeated symbols, because L and M are orthogonal; furthermore, because M is a Latin square, they do not repeat any rows or columns. Therefore, each symbol used in k corresponds to a different transversal of L, and we've proven our claim.

Lemma 3 *The circulant Latin square*

$$M = \begin{array}{|c|c|c|c|}\hline 0 & 1 & \dots & n-1 \\ \hline 1 & 2 & \dots & 0 \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline n-1 & 0 & \dots & n-2 \\ \hline \end{array}$$

does not have any transversals, if n is even.

Proof. Suppose that we had a transversal of M. Label its cells $\{(a_i, b_i)\}_{i=0}^{n-1}$; by construction, we know that the symbol contained within any cell (a_i, b_i) is $a_i + b_i$.

Then, on one hand, because this is a transversal, we know that every symbol occurs exactly once in our transversal, and therefore that

$$\begin{aligned} \sum_{i=0}^{n-1}(a_i + b_i) \equiv \sum_{i=0}^{n-1} i \equiv \frac{n(n-1)}{2} &\mod n \\ \equiv n\frac{n}{2} - \frac{n}{2} &\mod n \\ \frac{n}{2} &\mod n, \end{aligned}$$

where we used the fact that n is even to justify pulling the n out of the $\frac{n^2}{2}$ term. In particular, we know that this sum is not zero mod n.

However, if we look at our sum another way, we can instead think of it as the sum of all of the rows and all of the columns:

$$\begin{aligned} \sum_{i=0}^{n-1}(a_i + b_i) &\equiv \left(\sum_{i=0}^{n-1} a_i\right) + \left(\sum_{i=0}^{n-1} b_i\right) \mod n \\ &\equiv \left(\sum_{i=0}^{n-1} i\right) + \left(\sum_{i=0}^{n-1} i\right) \mod n \\ &\equiv \frac{n(n-1)}{2} + \frac{n(n-1)}{2} \mod n \\ &\equiv n(n-1) \mod n \\ &\equiv 0 \mod n. \end{aligned}$$

This is a contradiction! Therefore, no such transversal can exist.

Given this observation, we can now prove our claim from earlier:

Proposition. No pandiagonal Latin squares exist of even order.

Proof. Suppose that we had a pandiagonal Latin square L of even order. Consider the circulant Latin square

$M =$

0	1	$\dots$	$n-1$
1	2	$\dots$	0
$\vdots$	$\vdots$	$\ddots$	$\vdots$
$n-1$	0	$\dots$	$n-2$

We claim that M and L are orthogonal. To see this, simply notice that the broken left diagonals of M are just the same symbol repeated, while (by definition) the broken left diagonals of L contain every symbol exactly once. Therefore, when these two squares are superimposed, we will see every pair occur exactly once: to see any given pair, just pick the M-coordinate you're looking for, go to the broken diagonal that contains all of M's instances of that symbol, and search through the corresponding broken diagonal in L until you find the right L-coordinate.

But this is impossible, because L has no transversal while any square with an orthogonal mate contains a transversal, as we proved with our lemmas. Therefore, we cannot have a pandiagonal Latin square of order L.

This settles the case for even-order pandiagonal Latin squares. The only other case where we haven't either constructed a pandiagonal Latin square or shown they're impossible is when n is a multiple of 3: on your HW, there are a pair of questions that walk you through the proof that there is in fact **no pandiagonal Latin square with order divisible by 3**.

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Homework 6: Latin Squares and Chess

Week 3 — *Mathcamp 2012*

Attempt all of the problems that seem interesting, and let me know if you see any typos! (+) problems are harder than the others. (++) problems are currently open.

1. Construct a pandiagonal Latin square of order 7, and use it to solve the 7-queens problem.

2. While in class we said that pandiagonal Latin squares exist only when n is neither divisible by 2 or 3, it turns out that solutions to the n-queens problem exist for every value of $n \geq 4$. Using symmetry arguments, how many solutions can you find for $n = 5$? How about $n = 6$? (Hint: if you've done this problem correctly, there should be more solutions for $n = 5$ than $n = 6$. This is the only pair of numbers where this happens; in general, as n increases, the number of solutions to the n-queens problem grows exponentially, though it is an open question to how fast this exponential growth precisely is.)

3. Prove the claim we made in class, that there are no pandiagonal Latin squares with order divisible by 3. Do this via the following outline:

   (a) A **superdiagonal** of a $n \times n$ grid is a collection of n cells within this grid that contains exactly one representative from each row and column, as well as exacty one representative from each broken left diagonal and exactly one representative from each broken right diagonal.

   Show that a pandiagonal Latin square of order n exists if and only if it is possible to break the cells of a $n \times n$ grid up into n disjoint superdiagonals.

   (b) Show that a $n \times n$ array cannot have a superdiagonal if n is a multiple of 2 or 3.