Proof Techniques | Instructor: Padraic Bartlett

Homework 1

Week 1 | *Mathcamp 2012*

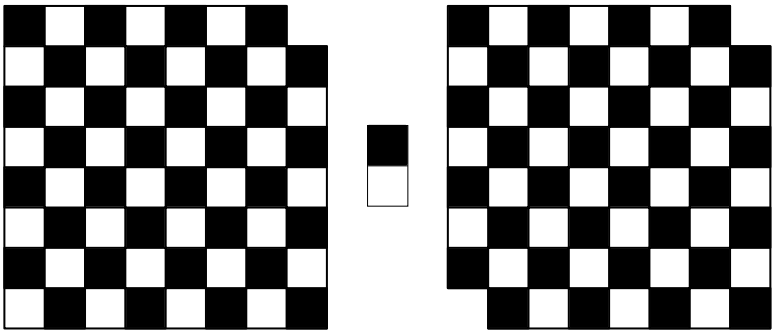
Homework instructions: many of the problems below are labeled with the tags $(*)$ or $(+)$. $(*)$ denotes that the problem in question is fairly fundamental to the topics we're studying and is something that you should make sure you understand completely, while $(+)$ denotes a problem that may be much harder than some of the others on the set.

This class is homework-required! What this means is that I'm expecting you to try every problem, to solve almost all of the $(*)$ ones, and most of the non-$(+)$ ones. The $(+)$ ones are certainly problems you are capable of solving, and I want you solve some of these! But they will not be as necessary for your ability to survive and thrive in later lectures, and I don't expect people to solve all of them. If you get stuck, or see a typo, find me! I can offer tons of hints and corrections. HW will be handed in at the start of class every week; I'll try to look over solutions in between classes, and come up with comments.

1. $[(*)]$ Given a real number x, let A be the statement "$\frac{11}{3} > x > \frac{5}{3}$", B be the statement "$x^2 = -1$," C be the statement $x^2 = 4$, and D be the statement $x \neq 2$. Which of the following statements are true for every $x \in \mathbb{R}$? Which are false for every $x \in \mathbb{R}$? Which are true for some values of x and false for other values of x? (Prove your answers.)

   (a) $C \Rightarrow A$.
   (b) $A \Rightarrow C$.
   (c) $B \vee ((\neg C) \wedge (\neg D))$.
   (d) $B \Rightarrow (C \wedge D)$.
   (e) $(A \vee B) \Rightarrow (\neg(C \vee D))$.
   (f) $D \Rightarrow \neg C$.
   (g) $C \Leftrightarrow D$.
   (h) $\neg(A \wedge B \wedge C) \Rightarrow (C \wedge D)$.

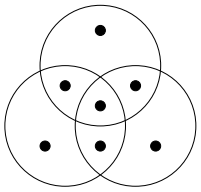
2. Take a 8×8 checkerboard and punch out its top-right corner (drawn below.) Can you completely cover it with 2×1 rectangles that don't overlap and don't hang off the board? What if you remove its top right and bottom-left corner; can you cover it with 2×1 rectangles then?



3. $[(*)]$ From outside of mathematics, come up with three statements A, B, C such that A and B together imply C, but neither A nor B alone are strong enough to imply C.

4. [(∗)] Show that the following statement is false: "If a and b are integers, then there are two integers m, n such that $a = m + n$ and $b = m - n$." What can be added to the hypothesis of this statement to make it true?

5. [(+)] Consider the following solitaire game:



The picture above contains three circles drawn in the plane. In each of the bounded regions formed by the intersections of these circles, we've placed a coin, which is white on one side and black on the other.

The moves you're allowed to perform in this game are the following:

- You can at any time flip all of the coins within any circle.
- Alternately, you can at any time take any circle and flip all of its white coins over to black.

Can you ever reach the following configuration? (Prove your claim.)