Proof Techniques Instructor: Padraic Bartlett

Lecture 3: Sizes of Infinity

Week 1 *Mathcamp 2012*

1 Sizes of Infinity

On one hand, we know that the real numbers contain "more" elements than the rational numbers: things like $\sqrt{2}$ are in $\mathbb{R}$ but not in $\mathbb{Q}$, for example. On the other hand, our "interleaving" result that we discussed on the HW (i.e. that between any two distinct real numbers there is a rational, and similarly between any two distinct rational numbers there is an irrational number) above seems to suggest that the sizes of these two sets might be somewhat similar: after all, if between any two real numbers there's a rational, how many "more" reals could you have?

In this section, we discuss how we can come up with a rigorous way of studying the above question. Let's start with the most basic thing we can ask: what does it mean for two sets to be the same size? In the finite case, this question is rather trivial; for example, we know that the two sets

$$A = \{1, 2, 3\}, \qquad B = \{A, B, \text{emu}\}$$

are the same size because they both have the same *number* of elements – in this case, 3.

But what about infinite sets? For example, look at the sets

$$\mathbb{N}, \quad \mathbb{Q}, \quad \mathbb{R}, \quad \mathbb{C};$$

are any of these sets the same size? Are any of them larger? By how much?

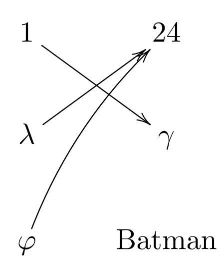
In the infinite case, the tools we used for the finite – counting up all of the elements – don't work. In response to this, we are motivated to try to find another way to count: in this case, one that involves **functions**.

1.1 Functions (formally defined)

Definition. A **function** f with domain A and range B, formally speaking, is a collection of pairs (a, b), with $a \in A$ and $b \in B$, such that there is exactly one pair (a, b) for every $a \in A$. More informally, a function $f : A \to B$ is just a map which takes each element in A to some element of B.

Examples.

- $f : \mathbb{Z} \to \mathbb{N}$ given by $f(n) = 2|n| + 1$ is a function.
- $g : \mathbb{N} \to \mathbb{N}$ given by $f(n) = 2|n| + 1$ is also a function. It is in fact a different function than f, because it has a different domain!
- The function h depicted below by the three arrows is a function, with domain $\{1, \lambda, \varphi\}$ and range $\{24, \gamma, \text{ Batman}\}$:

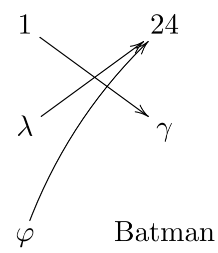


This may seem like a silly example, but it's illustrative of one key concept: functions are just **maps between sets!** Often, people fall into the trap of assuming that functions have to have some nice "closed form" like $x^3 - \sin(x)$ or something, but that's not true! Often, functions are either defined piecewise, or have special cases, or are generally fairly ugly/awful things; in these cases, the best way to think of them is just as a collection of arrows from one set to another, like we just did above.

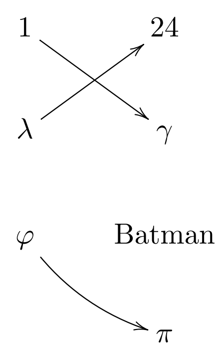
Now that we've formally defined functions and have a grasp on them, let's introduce a pair of definitions that will help us with our question of "size:"

Definition. We call a function f **injective** if it never hits the same point twice – i.e. for every $b \in B$, there is **at most one** $a \in A$ such that $f(a) = b$.

Examples. The function h from before is not injective, as it sends both λ and φ to 24:



However, if we add a new element π to our range, and make φ map to π, our function is now injective, as no two elements in the domain are sent to the same place:



One observation we can quickly make about injective functions is the following:

Proposition. If $f : A \to B$ is an injective function and A, B are finite sets , then $\text{size}(A) \leq \text{size}(B)$.

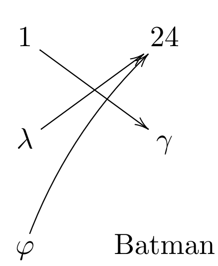
The reasoning for this, in the finite case, is relatively simple:

1. If f is injective, then each element in A is being sent to a different element in B.
2. Thus, you'll need B to have at least $|A|$-many elements to provide that many targets.

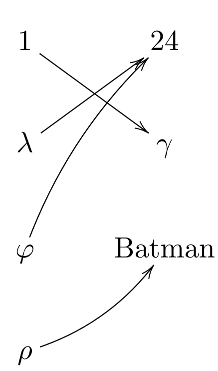
A converse concept to the idea of injectivity is that of **surjectivity**, as defined below:

Definition. We call a function f **surjective** if it hits every single point in its range – i.e. if for every $b \in B$, there is **at least one** $a \in A$ such that $f(a) = b$.

Examples. The function h from before is not injective, as it doesn't send anything to Batman:



However, if we add a new element ρ to our domain, and make ρ map to Batman, our function is now surjective, as it hits all of the elements in its range:



As we did earlier, we can make one quick observation about what surjective functions imply about the size of their domains and ranges:

Proposition. If $f : A \to B$ is an surjective function and A, B are finite sets , then $|A| \geq |B|$.

Basically, this holds true because

1. Thinking about f as a collection of arrows from A to B, it has precisely $|A|$-many arrows by definition, as each element in A gets to go to precisely one place in B.

2. Thus, if we have to hit every element in B, and we start with only $|A|$-many arrows, we need to have $|A| \geq |B|$ in order to hit everything.

So: in the finite case, if $f : A \to B$ is injective, it means that $|A| \leq |B|$, and if f is surjective, it means that $|A| \geq |B|$. This motivates the following definition and observation:

Definition. We call a function **bijective** if it is both injective and surjective.

Proposition. If $f : A \to B$ is an bijective function and A, B are finite sets , then $|A| = |B|$.

Unlike our earlier idea of counting, this process of "finding a bijection" seems like something we can do with any sets – not just finite ones! As a consequence, we are motivated to make this our **definition** of size! In other words, we have the following definition:

Definition. We say that two sets A, B are the same size (formally, we say that they are of the same **cardinality**,) and write $|A| = |B|$, if and only if there is a bijection $f : A \to B$.

1.2 The Natural Numbers

Armed with a definition of size that can actually deal with infinite sets, let's start with some calculations to build our intuition:

Question. Are the sets $\mathbb{N}$ and $\mathbb{N} \cup \{\text{lemur}\}$ the same size?

Answer. Well: we know that they can be the same size if and only if there is a bijection between one and the other. So: let's try to make a bijection! In the typed notes, the suspense is somewhat gone, but (at home) imagine yourself taking a piece of paper, and writing out the first few elements of $\mathbb{N}$ on one side and of $\mathbb{N} \cup \{\text{lemur}\}$ on the other side. After some experimentation, you might eventually find yourself with the following map:

$\mathbb{N}$		$\mathbb{N} \cup \{\text{lemur}\}$
1	$\to$	lemur
2	$\to$	1
3	$\to$	2
4	$\to$	3
5	$\to$	4
6	$\to$	5
$\vdots$		$\vdots$

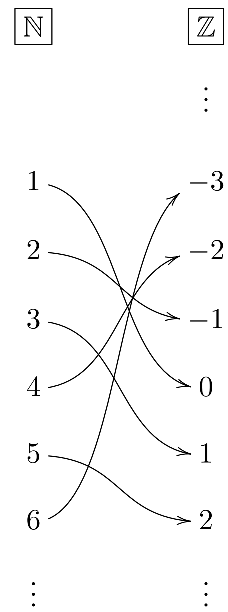
i.e. the map which sends 1 to the lemur and sends $n \to n - 1$, for all $n \geq 2$. This is clearly a bijection; so these sets are the same size!

In a rather crude way, we have shown that adding one more element to a set as "infinitely large" as the natural numbers doesn't do anything to it! – the extra element just gets lost amongst all of the others.

This trick worked for one additional element. Can it work for infinitely many? Consider the next proposition:

Proposition. The sets $\mathbb{N}$ and $\mathbb{Z}$ are the same cardinality.

Proof. Consider the following map:



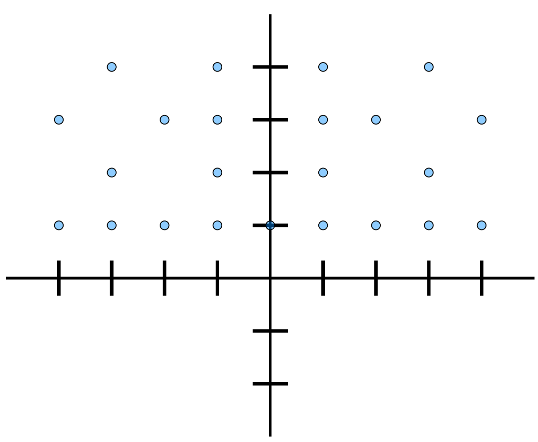
i.e. the map which sends $n \to (n-1)/2$ if n is odd, and $n \to -n/2$ if n is even. This, again, is clearly a bijection; so these sets are the same cardinality. □

So: we can in some sense "double" infinity! Strange, right? Yet, if you think about it for a while, it kind of makes sense: after all, don't the natural numbers contain two copies of themselves (i.e.the even and odd numbers?) And isn't that observation just what we used to turn $\mathbb{N}$ into $\mathbb{Z}$?

After these last two results, you might be beginning to feel like all of our infinite sets are the same size. In that case, the next result will hardly surprise you:

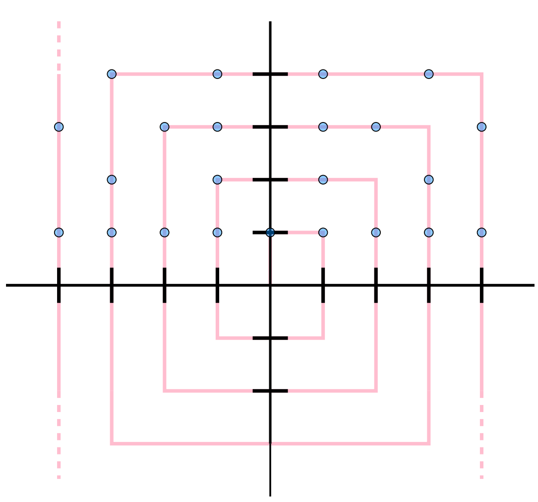
Proposition. The sets $\mathbb{N}$ and $\mathbb{Q}$ are the same cardinality.

Proof. First, take every rational number p/q with $GCD(p,q) = 1$, $p > 0$, and draw a point at (p,q) in the integer lattice $\mathbb{Z}^2$:



In the picture on the previous page, every rational number has exactly one unique representative by one of our blue dots.

Now, on this picture, draw a spiral that starts at $(0,0)$ and goes through every point of $\mathbb{Z} \times \mathbb{Z}$, as depicted below:



We use this spiral to define our bijection from $\mathbb{N}$ to $\mathbb{Q}$ as follows:

$f(n) =$ *the n-th rational point found by starting at (0,0) and walking along the depicted spiral pattern.*

This function hits every rational number exactly once by construction; thus, it is a bijection from $\mathbb{N}$ to $\mathbb{Q}$. Consequently, $\mathbb{N}$ and $\mathbb{Q}$ are the same size. □

1.3 The Reals

At this point, it almost seems inevitable that **every** infinte set will wind up having the same size!

This is false.

Theorem. The sets $\mathbb{N}$ and $\mathbb{R}$ have different cardinalities.

Proof. (This is Cantor's famous diagonalization argument.) Suppose not – that they were the same cardinalities. As a result, there is a bijection between these two sets! Pick such a bijection $f : \mathbb{N} \to \mathbb{R}$.

For every $n \in \mathbb{N}$, look at the number $f(n)$. It has a decimal representation. Pick a number $a_{n,\text{trash}}$ corresponding to the integer part of $f(n)$, and $a_{n\_1}, a_{n\_2}, a_{n\_3}, \ldots$ that correspond to the digits after the decimal place of this decimal representation – i.e. pick numbers a_{n_i} such that

$$f(n) = a_{n\_\text{trash}}.a_{n\_1}a_{n\_2}a_{n\_3}\ldots$$

For example, if $f(4) = 31.125$, we would pick $a_{4\_\text{trash}} = 31, a_{4\_1} = 1, a_{4\_2} = 2, a_{4\_3} = 5$, and $0 = a_{4\_4} = a_{4\_5} = a_{4\_6} = \ldots$, because the integer part of $f(4)$ is 31, its first three digits after the decimal place are 1,2, and 5, and the rest of them are zeroes.

Now, get rid of the a_{n_trash} parts, and write the rest of these numbers in a table, as below:

$f(1)$	a_{1_1}	a_{1_2}	a_{1_3}	a_{1_4}	$\ldots$
$f(2)$	a_{2_1}	a_{2_2}	a_{2_3}	a_{2_4}	
$f(3)$	a_{3_1}	a_{3_2}	a_{3_3}	a_{3_4}	
$f(4)$	a_{4_1}	a_{4_2}	a_{4_3}	a_{4_4}	
$\vdots$	$\vdots$				$\ddots$

In particular, look at the entries $a_{1\_1}a_{2\_2}a_{3\_3}\ldots$ on the diagonal. We define a number B using these digits as follows:

- Define $b_i = 2$ if $a_{i\_i} \neq 2$, and $b_i = 8$ if $a_{i\_i} = 2$.
- Define B to the be the number with digits given by the b_i – i.e.

$$B = .b_1b_2b_3b_4\ldots$$

Because B has a decimal representation, it's a real number! So, because our function f is a bijection, it must have some value of n such that $f(n) = B$. But the n-th digit of $f(n)$ is $a_{n,n}$ by construction, and the n-th digit of B is b_n – by construction, these are different numbers! So $f(n) \neq B$, because they disagree at their n-th decimal place!

This is a contradiction to our original assumption that such a bijection existed. Therefore, we know that no such bijection can exist: as a result, we've shown that the natural numbers are of a strictly "smaller" size of infinity than the real numbers. □

Crazy.