Lecture 3: The Concept of Quasirandomness

1 Defining Quasirandomness

Something odd you may have noticed in yesterday's lecture is the following: when we were proving that the eigenvalues $\lambda_1, \ldots \lambda_n$ of a random graph's adjacency matrix were of the form $\lambda_1 \cong \frac{n}{2}, \lambda_2, \ldots \lambda_n = o(n)$, we really never actually directly worked with a random adjacency matrix! Instead, the only two facts we used were these two:

- The total number of expected 4-cycles in a random graph is roughly $\leq \frac{n^4}{16}$.
- The expected degree of every vertex in a random graph is roughy $\geq \frac{n}{2}$.

By using these two observations, we were able to get an upper bound on the sum of the fourth powers of the eigenvalues and a lower bound on λ_1, respectively; combining these two inequalities gave us our claimed result.

This observation, that knowing that a graph satisfied two kinds of "randomness" properties forced it to have certain other kinds of randomness properties, is the motivation for the following definition, and indeed this entire class!

Definition. Let $\mathcal{G} = \{G_{k_n}\}_{n=1}^{\infty}$ be any sequence of graphs, each on k_n vertices, where the k_n's are an nondecreasing sequence that tends to infinity. We say that this sequence is **quasirandom** if, roughly speaking, it "looks like a random graph" in a number of quantifiable ways.

To make this rigorous, here's an additional bit of notation. Suppose that G, H are two graphs. Let $N^*_G(H)$ denote the number of labelled occurences of H as an induced subgraph of G. Similarly, let $N_G(H)$ denote the number of labeled occurences of H as a subgraph of G (not necessarily induced.)

We say that our sequence $\mathcal{G}$ is quasirandom if and only if its elements satisfy the following list of asymptotic properties, as the number of vertices in any such element G goes to infinity:

$P_1(s)$: For any graph H_s on s vertices,

$$N^*_G(H_s) = (1 + o(1)) \cdot n^s \cdot 2^{-\binom{s}{2}}.$$

$P_2(t)$: Let C_t denote the cycle of length t. Then

$$e(G) \geq (1 + o(1)) \cdot \frac{n^2}{4}, \text{ and}$$
$$N_G(C_t) \leq (1 + o(1)) \cdot \frac{n^t}{2^t}.$$

P_3: Let $A(G)$ denote the adjacency matrix of G, and $|\lambda_1| \geq \ldots \geq |\lambda_n|$ be the eigenvalues of $A(G)$. Then

$$e(G) \geq (1 + o(1)) \cdot \frac{n^2}{4}, \text{ and}$$
$$\lambda_1 = (1 + o(1)) \cdot \frac{n}{2}, \quad \lambda_2 = o(n).$$

P_4: Given any subset $S \subseteq V$,

$$e(S) = \frac{|S|^2}{4} + o(n^2).$$

P_5: Given any pair of vertices $v, v' \in G$, let $s(v, v')$ denote the number of vertices y such that both (v, y) and (v', y) are either both edges or both nonedges in G. Then

$$\sum_{v,v'} \left| s(v, v') - \frac{n}{2} \right| = o(n^3).$$

In addition, a useful property to note is the "P_0" property:

P_0:

$$\sum_{v \in V} \left| \deg(v) - \frac{n}{2} \right| = o(n^2).$$

Another way to phrase P_0 is as the following claim:

P_0': All but $o(n)$ vertices in G have degree $(1 + o(1))\frac{n}{2}$.

(HW problem for this class: prove that these are equivalent, if you don't believe this!) The P_0 property is strictly weaker than any of the above properties: another HW problem is to show that P_0 is implied by any of the 5 above properties.

In the definition above, we said that a graph has to satisfy **all** of the properties above in order to be quasirandom. However, as we saw yesterday, some of these properties are related: specifically, we saw yesterday that $P_2(4)$ and P_0 — i.e. knowing the number of 4-cycles and the degrees — was enough to show that P_3 holds.

As it turns out, this equivalence property holds for **all** of the properties above! In other words, we have the following theorem:

Theorem. Suppose that $\mathcal{G}$ is a sequence of graphs that satisfies any one of

- $P_1(s)$, for some $s \geq 4$, or
- $P_2(t)$, for some $t \geq 4$, or
- P_3, or P_4, or P_5.

Then it satisfies all of these properties. In other words, all of these properties are equivalent!

This is fairly surprising: at first glance, it doesn't seem like having the right number of 4-cycles should be enough to force you to have the right number of **any** induced subgraph (like, say, the right number of copies of the Petersen graph!) And yet, this turns out to be true. Today and tomorrow's lectures will be devoted to proving this resut.

2 The Massive Equivalence Proof, 1/2

A map of our proof is the following:

$$\begin{array}{ccccccccccc}
P_1(s+1) & \Rightarrow & P_2(s+1) & & & & & & & & \\
\Downarrow & & \Downarrow & & & & & & & & \\
P_1(s) & \Rightarrow & P_2(s) & & & & & & & & \\
\Downarrow & & \Downarrow & & & & & & & & \\
\vdots & & \vdots & & & & & & & & \\
P_1(4) & \Rightarrow & P_2(4) & \Rightarrow & P_3 & \Rightarrow & P_4 & \Rightarrow & P_5 & \Rightarrow & P_1(t), \text{ for all } t.
\end{array}$$

A part of our proof that we'll be using but not proving in class is that P_0 is implied by all of our properties: showing this is part of the HW over the next two days, and you should do it if you feel like the following proofs will feel pointless without it.

We start here. Throughout these proofs, assume that G is a n-vertex graph that comes from a sequence $\mathcal{G}$, that we're trying to show is quasirandom.

Proposition. $P_1(s+1) \Rightarrow P_1(s)$.

Proof. Take any graph $M(s)$ on s vertices, and make the following two observations:

1. There are 2^s-many ways (counting different labelings as distinct) to extend any such $M(s)$ to a labeled graph on $s+1$ vertices. To see this, consider the process of adding a $n+1$-th vertex v to $M(s)$: for every vertex w in $M(s)$, we'll have decide whether (v, w) is or is not an edge, which accounts for our 2^s many choices.

2. As well, if you take any copy of $M(s)$ in G, there are precisely $n-s$ induced subgraphs on $s+1$ vertices of G that contain that $M(s)$ as a subgraph (as any such graph is formed by simply choosing another vertex of G.

By combining these observations, we can derive the following relation between $N^*_G(M(s+1))$ and $N^*_G(M(s))$.

$$\frac{N^*_G(M(s)) \cdot (n-s)}{2^s} = N^*_G(M(s+1)).$$

Therefore, if $P_1(s+1)$ holds, we can use the property that

$$N^*_G(M(s+1)) = (1+o(1)) \cdot n^{s+1} \cdot 2^{-\binom{s+1}{2}}$$

to deduce that

$$\begin{aligned}
& \frac{N^*_G(M(s)) \cdot (n-s)}{2^s} = (1+o(1)) \cdot n^{s+1} \cdot 2^{-\binom{s+1}{2}} \\
\Rightarrow \quad & N^*_G(M(s)) = (1+o(1)) \cdot n^s \cdot 2^{-\binom{s}{2}}.
\end{aligned}$$

This is precisely $P_1(s)$.

On the HW, you're asked to prove the following:

Proposition. $P_1(3) \Rightarrow P_0$.

Combining these two results tells us that $P_1(s) \Rightarrow P_0$, for any $s \geq 3$: i.e. that if we have the right number of all induced subgraphs on s vertices, we also have all of our vertices with about the right degree. Which seems reasonable.

Using this, we can move on to our next result:

Proposition. $P_1(t) \Rightarrow P_2(t)$, for $t \geq 3$.

Proof. First, notice that because $P_1(t) \Rightarrow P_1(t-1) \Rightarrow \ldots \Rightarrow P_1(3) \Rightarrow P_0$, we immediately have the edge condition

$$e(G) \geq (1 + o(1)) \cdot \frac{n^2}{4}$$

for P_2. So it suffices to verify that we also have the "right number" of t-cycles.

Take any cycle C_t with labeled vertices. There are precisely $2^{\binom{t}{2}-t}$-many ways to extend this cycle to a subgraph where we've determined whether every edge either exists or does not exist: i.e. there are $2^{\binom{t}{2}-t}$-many different labeled graphs whose existence as labeled induced subgraphs could correspond to this specific cycle.

Therefore, we have that

$$N_G(C_t) = \sum_H N^*_G(H) = 2^{\binom{t}{2}-t} \cdot (1 + o(1)) \cdot n^t \cdot 2^{-\binom{t}{2}} = (1 + o(1)) \cdot \frac{n^t}{2^t}.$$

This is precisely $P_2(t)$.

Proposition. $P_2(2k) \Rightarrow P_3$.

Proof. For $2k = 4$, this is exactly what we did yesterday when we were studying the eigenvalues of a random graph: instead of directly calculating them, we instead found them by using the number of 4-cycles!

It bears noting that the proof for $2k$-cycles is pretty much identical: instead of looking at A^4, you look at A^{2k}, and you will wind up with the same inequalities after taking $2k$-th roots. Persuade yourself of this fact if you're skeptical!

Proposition. $P_3 \Rightarrow P_4$.

Proof. Let $A(G)$ be the associated adjacency matrix to the graph G, $|\lambda_1| \geq \ldots \geq |\lambda_n|$ be its eigenvalues, and $\mathbf{e}_1, \ldots \mathbf{e}_n$ the corresponding eigenvectors (via the spectral theorem, pick these so that they're all orthogonal to each other and also all of length 1.) As well, let $\mathbf{u} = \left(\frac{1}{\sqrt{n}}, \ldots \frac{1}{\sqrt{n}}\right)$.

We claim first that the corresponding eigenvector $\mathbf{e}_1$ of to λ_1 A is "roughly" $\mathbf{u}$: i.e. that $||\mathbf{u} - \mathbf{e}_1||$ is $o(1)$. To see this, simply write $\mathbf{u} = \sum_{j=1}^n a_j \mathbf{e}_j$: we can do this because the $\mathbf{e}_j$'s form a basis for $\mathbb{R}^n$! (yay spectral theorem!)

Then, on one hand, we have that

$$A\mathbf{u} = A \cdot \left(\sum_{j=1}^{n} a_j \mathbf{e}_j \right) = \sum_{j=1}^{n} a_j \lambda_j \mathbf{e}_j.$$

On the other, we also have that $A\mathbf{u} = \frac{1}{\sqrt{n}} (\deg_i(v_1), \ldots \deg_i(v_n))$. Because $P_3 \Rightarrow P_0$, we know that all but $o(n)$ of these vertices have degree $(1+o(1))\frac{n}{2}$; therefore, we know that we can write

$$A\mathbf{u} = \left((1+o(1))\frac{n}{2} \right) \cdot \mathbf{u} + \mathbf{w},$$

for some vector $\mathbf{w}$ with all but $o(n)$ of its components with magnitude $o(\sqrt{n})$. This forces $||\mathbf{w}|| = o(n)$. Now, if we think about what this means for the eigenvalues of A, we can use P_3 to show that

$$\begin{aligned}
&\sum_{j=1}^{n} a_j \lambda_j \mathbf{e}_j = \left((1+o(1))\frac{n}{2} \right) \cdot \mathbf{u} + \mathbf{w} \\
\Rightarrow &\sum_{j=1}^{n} a_j \left(\lambda_j - \frac{n}{2} \right) \mathbf{e}_j = o(1) \cdot \frac{n}{2} \cdot \mathbf{u} + \mathbf{w} \\
\Rightarrow &\left|\left| \sum_{j=1}^{n} a_j \left(\lambda_j - \frac{n}{2} \right) \mathbf{e}_j \right|\right| = \left|\left| o(1) \cdot \frac{n}{2} \cdot \mathbf{u} + \mathbf{w} \right|\right| = o(n) \\
\Rightarrow &\left(\sum_{j=1}^{n} a_j^2 \left(\lambda_j - \frac{n}{2} \right)^2 \right)^{1/2} = o(n) \\
\Rightarrow &\left(\sum_{j=2}^{n} a_j^2 \left(\frac{n}{2} \right)^2 \right)^{1/2} = o(n) \\
\Rightarrow &\sum_{j=2}^{n} a_j^2 = o(1).
\end{aligned}$$

Therefore, we have that $\mathbf{u} = a_1\mathbf{e}_1 + \mathbf{v}$, for some vector $\mathbf{v}$ with $||\mathbf{v}|| = o(1)$. This tells us that $|a_1| = 1 + o(1)$.

Now, we need to use a very very large hammer from linear algebra, called the Perron-Frobenius theorem, to note the following: because $\mathbf{e}_1$ is the eigenvector corresponding to the largest eigenvalue of a nonnegative symmetric matrix A, the Perron-Frobenius theorem says that all of the entries in the vector $\mathbf{e}_1$ are nonnegative. Accept this on faith, because the Perron-Frobenius theorem could easily be a 3-4 chili class in its own right!

If we use this on faith, we have just shown that $a_1 = 1 + o(1)$. This proves our claim that $||\mathbf{u} - \mathbf{e}_1||$ is $o(1)$.

Let us use this fact in proving our current proposition. Given any subset $S \subseteq V$, set χ_S to be the characteristic vector of S: i.e. χ_S has a 1 in its j-th slot if $v_j \in S$, and a 0 otherwise. As well, define $\mathbf{s} = \chi_S - \langle \chi_S, \mathbf{e}_1 \rangle \mathbf{e}_1$: i.e. $\mathbf{s}$ is the result of taking χ_1 and subtracting off its $\mathbf{e}_1$-component.

With these definitions made, let us examine the quantity $\langle A\mathbf{s}, \mathbf{s} \rangle$ in two different ways. On one hand, if we use our claim from earlier, we have

$$\begin{aligned}
\langle A\mathbf{s}, \mathbf{s} \rangle &= \langle A(\chi_S - \langle \chi_S, \mathbf{e}_1 \rangle \mathbf{e}_1), \chi_S - \langle \chi_S, \mathbf{e}_1 \rangle \mathbf{e}_1 \rangle \\
&= \langle A\chi_s, \chi_s \rangle - \langle A\chi_s, \langle \chi_s, \mathbf{e}_1 \rangle \mathbf{e}_1 \rangle - \langle A \langle \chi_s, \mathbf{e}_1 \rangle \mathbf{e}_1, \chi_s \rangle + \langle A \langle \chi_s, \mathbf{e}_1 \rangle \mathbf{e}_1, \langle \chi_s, \mathbf{e}_1 \rangle \mathbf{e}_1 \rangle \\
&= \langle A\chi_s, \chi_s \rangle - \lambda_1 \langle \chi_s, \mathbf{e}_1 \rangle^2 \\
&= 2e(S) - \lambda_1 \langle \chi_s, \mathbf{e}_1 \rangle^2 \\
&= 2e(S) - \lambda_1 \langle \chi_s, \mathbf{u} + \mathbf{v} \rangle^2 \\
&= 2e(S) - \lambda_1 \left(\frac{|S|}{\sqrt{n}} + o(\sqrt{|S|} \right)^2 \\
&= 2e(S) - \left(\frac{1}{2} + o(1) \right) |S|^2 + o(n^2).
\end{aligned}$$

On the other, if we use the observation that $\mathbf{s}$ is orthogonal (by construction) to $\mathbf{e}_1$, we can see that

$$\langle A\mathbf{s}, \mathbf{s} \rangle \leq |\lambda_2| \cdot ||\mathbf{s}||^2 = |\lambda_2| \cdot ||\chi_s - \langle \chi_S, \mathbf{e}_1 \rangle \mathbf{e}_1||^2 \leq ||\chi_S|| = |\lambda_2| \cdot |S| = o(n) \cdot |S|.$$

Combining these two observations tells us that

$$\begin{aligned}
2e(S) - \left(\frac{1}{2} + o(1) \right) |S|^2 + o(n^2) &\leq o(n) \cdot |S| \\
\Rightarrow e(S) &= (1 + o(1)) \frac{|S|^2}{4}.
\end{aligned}$$

This is precisely P_4.