Random and Quasirandom Graphs	Instructor: Padraic Bartlett

Lecture 5: More Examples/Applications of Quasirandom Graphs

Week 4 — *Mathcamp 2012*

In our last class, we saw that the Paley graphs were an example sequence of quasirandom graphs. In this lecture, we will explore two more classes of quasirandom graphs, and talk about what quasirandomness tells us about these two classes.

1 Quasirandom Binary Sequences

For our first example, consider the following family of graphs:

Definition. Let k be an odd integer. Define H_k as the following graph:

- Our vertex set is the collection of all vectors of length k, made of 0's and 1's, so that there is an odd number of 1's. Remove the all-1's vector from this list (you don't have to do this, but it makes the calculations super-trivial.)
- We connect two vertices with an edge if and only if the dot product of their two corresponding vectors is an odd number. For example,

  $$(0, 1, 1, 0, 1) \cdot (1, 0, 0, 1, 1) = 1,$$

  and therefore we would connect these with an edge, but

  $$(0, 1, 1, 0, 1) \cdot (1, 0, 0, 0, 0) = 0,$$

  so we would not connect these with an edge.

We claim that this sequence of graphs is quasirandom. To see why, note that if we believe our claim from yesterday — that all of the quasirandom graph properties are equivalent — then we can simply show that this graph satisfies one of our properties to see that it satisfies all of them. In particular, we can just show that this graph satisfies P_5 (we pick this property because it's pretty easy to check, unlike the "all subgraphs occurring with the same frequency" property.)

To do this, take any two vectors $\mathbf{v}, \mathbf{w}$ from our set H_k. How many vectors $\mathbf{x}$ exist such that either both $(\mathbf{v}, \mathbf{x}), (\mathbf{w}, \mathbf{x})$ are edges, or both are non-edges? Well: this happens if and only if these two dot products are both odd or both even: i.e. if

$$\mathbf{v} \cdot \mathbf{x} \equiv \mathbf{w} \cdot \mathbf{x} \mod 2.$$

How many ways are there to do this? Well: in the places where $\mathbf{v}, \mathbf{w}$ agree, the values of $\mathbf{x}$ don't affect whether these two dot products share the same parity, because they change the left and right hand sides equally.

In the places where $\mathbf{v}, \mathbf{w}$ disagree, however, each coordinate of $\mathbf{x}$ that is 1 changes the parity by 1. Therefore, we must have an even number of these coordinates that are 1, if our parity is the same on both sides. This in turn means that we have to have an odd number of 1-coordinates on the places where $\mathbf{v}, \mathbf{w}$ agree, so that the total number of 1's is odd.

Furthermore, note that neither of these coordinate sets are empty: these vectors must disagree in some places (otherwise, they're the same) and agree in others (otherwise, their dot product is 0, which is not $1 \mod 2$.)

So: without any loss of generality, for some value m between 1 and $k-1$ we're looking to count the total number of 0-1 vectors of length k, such that the following holds: in their first m coordinates there is an odd number of 1's, and in their remaining $k-m$ coordinates there's an even number of 1's. If you simply think of breaking this into two vectors, one of length k, the other of length $m-k$, it should be clear that this count is precisely a quarter of all of the possible vectors: the first condition eliminates half of the possible vectors from consideration, and the second condition eliminates another half. If we also take into account that we aren't considering $\mathbf{v}, \mathbf{w}$ as possible choices for $\mathbf{x}$, this means that there are precisely

$$\frac{\text{number of vectors of length } k}{4} - 2 = \frac{2^k}{4} - 2 = 2^{k-2} - 2$$

such values of $\mathbf{x}$.

There are 2^{k-1} many vectors of length k with an odd number of 1's in total: therefore, if we go to calculate the sum for P_5, we would have

$$\begin{aligned}\sum_{\mathbf{v},\mathbf{w}} \left| s(\mathbf{v},\mathbf{w}) - \frac{2^{k-1}}{2} \right| &= \sum_{\mathbf{v},\mathbf{w}} \left| 2^{k-2} - 2 - \frac{2^{k-1}}{2} \right| \\ &= \sum_{\mathbf{v},\mathbf{w}} 2 = 2(\#\text{ vertex pairs}) = o((\#\text{ vertices})^3).\end{aligned}$$

So this graph is quasirandom.

2 Turning Affine Planes into Projective Planes into Quasirandom Graphs

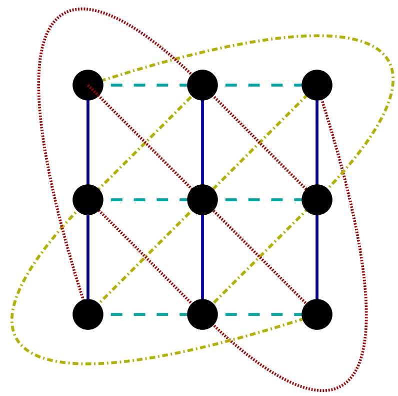
Definition. An **affine plane** is a collection of points and lines in space that follow the following fairly sensical rules:

(A1): Given any two points, there is a unique line joining any two points.

(A2): Given a point P and a line L not containing P, there is a unique line that contains P and does not intersect L.

(A3): There are four points, no three of which are collinear. (This rule is just to eliminate the silly case where all of your points are on the same line.)

$\mathbb{R}^2$ satisfies these properties, and as such is an affine plane. In this last bit of class, we're going to be interested in studying **finite** affine planes: i.e. affine planes with finitely many points. For example, the following set of nine points and twelve lines defines an affine plane:



Affine planes satisfy a ton of properties:

Proposition. In any affine plane, there is an integer n such that every line in our plane contains exactly n points, and every point lies on precisely $n+1$ lines. (We call this value the **order** of our plane.)

Any finite affine plane of order n contains n^2 many points. Any affine plane of order n contains exactly n^2+n lines in this plane; these lines can be partitioned into $n+1$ collections, each of which contains n parallel lines.

Affine planes of order n exist for any n that is a prime power. It is currently unknown whether these exist for other values of n.

For a proof of these properties, as well as a discussion about how we can generate these affine planes using Latin squares, read the notes on Latin squares from my class earlier in camp!

Instead, what I want to discuss is how to turn an affine plane into a **projective plane**:

Definition. A **projective plane** is a different geometric object, formed by the following three rules:

(P1): Given any two points, there is a unique line joining any two points.

(P2): Any two distinct lines intersect at a unique point.

(P3): There are four points, no three of which are collinear. (Again, this rule eliminates the silly case where all of your points are on the same line.)

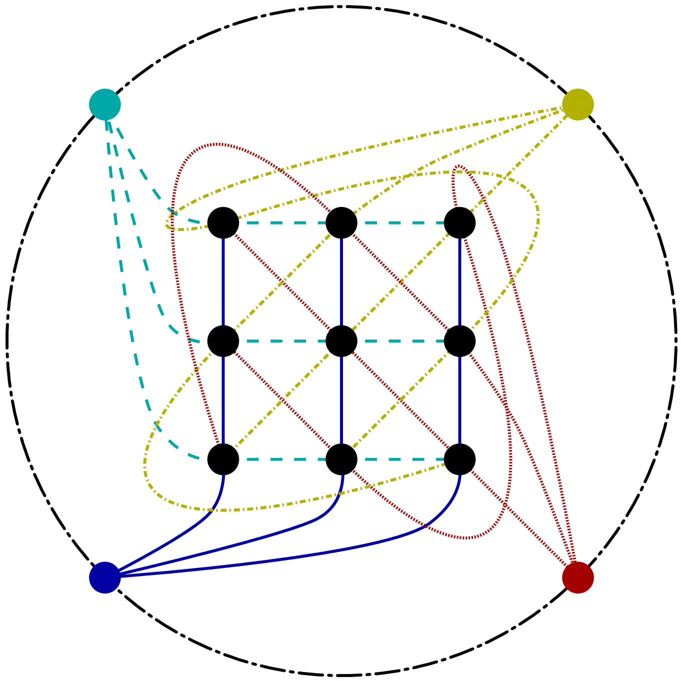
Basically, we have replaced our earlier axiom of ''Given any line L and point P, there is exactly one line parallel to L through P" with the axiom "There are no parallel lines."

You can turn an affine plane of order n into a projective plane of order n, and vice-versa, via the following construction:

- Take an affine plane, and split its lines into $n+1$ collections of n parallel lines. Label these collections $C_1, \ldots C_{n+1}$.
- For each class C_i, add a point ∞_i, and extend every line in this class C_i to contain this point ∞_i.

- Create a line L_∞ that consists of all of these points $\{\infty_i\}_{i=1}^{n+1}$.

We illustrate the results of this process below:



We mention these geometric objects because (surprisingly!) we can turn them into a quasirandom graph! We do this as follows:

- Start with an affine plane A of order n, and augment it via the construction above into a projective plane A'. Let L_∞ be the line "at infinity" created by the above process.
- Take the line L_∞, and split its points up into two disjoint sets N_+, N_-, such that each of these sets contains half of the points at on this line at infinity.
- Turn this into a graph as follows: our vertex set is the n^2-many points in our affine plane A.
- Connect two vertices x, y in our vertex set with an edge $\{x, y\}$ if and only if the following happens: take the unique line in our affine plane A through these two points x, y. If it goes through one of the ∞_i-points in our set N_+, draw an edge connecting $\{x, y\}$; instead, if it goes through a point in N_-, do not draw an edge.

This defines a graph! To see that this forms a quasirandom graph, we check P_5, just like in our earlier examples: take any pair of vertices x, y. We want to count how many vertices z are such that either both $\{x, z\}, \{y, z\}$ are edges, or $\{x, z\}, \{y, z\}$ are both not edges.

To do this, take any point ∞_i on our line at infinity, and draw the line connecting x to ∞_i. Because A was an affine plane before our construction, there are $n-1$ points $z_1, \ldots z_{n-1}$ on this extended line that are neither x or ∞_i.

Take the vertex y, and look at the $n-1$ lines formed by starting at y and drawing a line from y through each of the $n-1$ points on the extended line $\overline{x\infty_i}$. Because none of these lines are parallel, each of them goes through a different point at infinity: therefore, half of the possible values z_j go through points in N_-, while the other half go through points in N_+.

This is true for **any** point ∞_i at infinity! In particular, for any point ∞_i in N_+, half of our choices of z go through a point in N_+, and for any point in ∞_i **not** in N_-, half of our choices of z go through a point in N_-. Therefore, in total, given a pair of points x, y, half (up to rounding) of the possible third choices of z are such that either both $\{x, z\}, \{y, z\}$ are edges, or $\{x, z\}, \{y, z\}$ are both not edges.

This means that the sum

$$\sum_{\mathbf{x},\mathbf{y}} \left| s(\mathbf{x}, \mathbf{y}) - \frac{n^2}{2} \right|$$

is definitely $o(n^3)$; therefore that we satisfy P_5, and therefore this graph is quasirandom.