

Lecture 1: The Unit Distance Graph and the Axiom of Choice

Week 5

Mathcamp 2012

1 The Unit Distance Graph Problem

Question. Consider the following method for turning \mathbb{R}^2 into a graph:

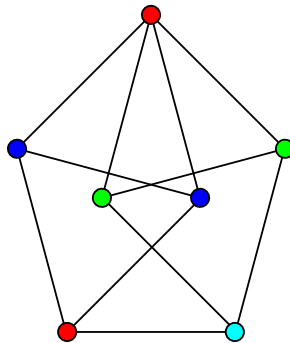
- Vertices: all points in \mathbb{R}^2 .
- Edges: connect any two points (a, b) and (c, d) iff the distance between them is exactly 1.

What is the chromatic number¹ of this graph?

To get a lower bound, we can consider graphs that we can draw in the plane such that all of the edges have length 1. Specifically, given any such graph G , we know that $\chi(G) \leq \chi(\mathbb{R}^2)$, because \mathbb{R}^2 contains any such graph as a subgraph! Therefore, by examining an equilateral triangle T , for example, we can see that

$$\chi(\mathbb{R}^2) \geq 3,$$

because there's no way to color an equilateral triangle with less than three colors so that no edge has both endpoints with the same color. Similarly, by examining the following pentagonal construction (called a Moser spindle,)



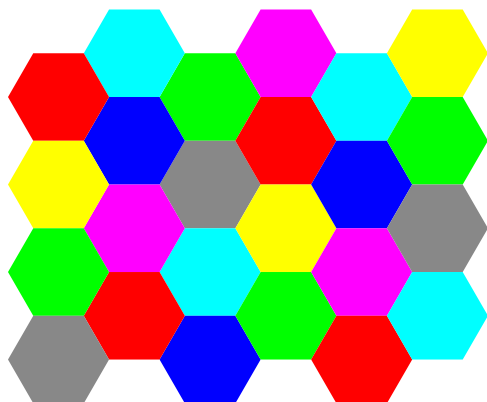
we can actually do one better and say that

$$\chi(\mathbb{R}^2) \geq 4,$$

because this graph cannot be properly colored with any less than 4 colors.

Conversely: to exhibit an upper bound on $\chi(\mathbb{R}^2)$ of k , it suffices to create a way of “painting” the plane with k -colors in such a way that no two points distance 1 apart get the same color. For example, look at the following way to paint the plane with hexagons of seven different colors, where each hexagon has diameter slightly less than 1:

¹The **chromatic number** of a graph G is the smallest number of colors you need to assign a color to every vertex on your graph, so that no edge has two endpoints of the same color.



In the above picture, no two hexagons with the same color are adjacent, or within one hexagon of each other; therefore, this is a proper coloring of the unit plane with seven colors! So, we have

$$\chi(\mathbb{R}^2) \leq 7.$$

These bounds on $\chi(\mathbb{R}^2)$ took us a little more than a page to accomplish; as a result, we might hope that completely resolving this question is something we could easily finish within the lecture! (After all, it's a relatively simple question: how hard can it be to find the chromatic number of the plane, anyways?)

As it turns out: really, really hard. This problem – often called the Hadwiger-Nelson problem in graph theory literature – has withstood attacks from the best minds in combinatorics since the 1950's. In today's lecture, we're going to talk briefly about how this result may (surprisingly) depend on the Axiom of Choice!

We start by first saying what the axiom of choice is:

2 Axioms in Mathematics

On yesterday's problem set, we defined the following:

Axiom of Choice For every family Φ of nonempty sets, there is a choice function

$$f : \Phi \rightarrow \bigcup_{S \in \Phi} S,$$

such that $f(S) \in S$ for every $S \in \Phi$.

Roughly speaking, this axiom says that for any collection of sets, we can pick one element out of each set.

In the early 1960's, Paul Cohen and Kurt Gödel proved that the axiom of choice is independent of the Zermelo-Fraenkel axioms of set theory, the current framework within which we do mathematics: i.e. that it is its own proper axiom! Pretty much all of modern mathematics accepts the Axiom of Choice; it's an incredibly useful axiom, and most fields of mathematics need to be able to call on it when pursuing nonconstructive proofs.

There are, however, a number of disconcerting “paradoxes” that arise from working within ZFC, the framework of axioms given by the Zermelo-Fraenkel axioms + the axiom of choice:

- The well-ordering principle: the statement that any set S admits a well-ordering². One consequence of this is that there's a way to order the real numbers so that they "locally" look like the natural numbers! Which is strange.
- The Banach-Tarski paradox: there's a way to chop up and rearrange a sphere into two spheres of the same surface area. Which is also strange.
- The existence of nonmeasurable sets: There are bounded subsets of the real line to which we cannot assign any notion of "length," given that we want length to be a translation-invariant, nontrivial, and additive function on subsets of \mathbb{R} . Which is also also strange.

Motivated by these strange results, Solovay (a set theorist) introduced the following two "alternate" axioms to choice:

- (AC_{\aleph_0} , the countable axiom of choice): For every **countable** family Φ of nonempty sets, there is a choice function

$$f : \Phi \rightarrow \bigcup_{S \in \Phi} S,$$

such that $f(S) \in S$ for every $S \in \Phi$.

- (LM, Lebesgue-measurability): Every bounded set in \mathbb{R} is measurable³.

From here, he proved that there are models of mathematics in which we can use and study these axioms in place of AC :

Theorem 1 (Solovay's Theorem) *There are models of mathematics in which $ZF + LM + AC_{\aleph_0}$ all hold.*

For brevity's sake, we will denote $ZF +$ the axiom of choice by ZFC , and $ZF + LM + AC_{\aleph_0}$ by ZFS .

3 $\chi(\mathbb{R}^2)$ in ZFS

At the start of lecture, I claimed that $\chi(\mathbb{R}^2)$ may depend on AC . The "may" in that statement comes from the fact that we currently don't know what $\chi(\mathbb{R}^2)$ **is**, and therefore it's difficult to say what it depends on. However, consider the following graph G :

²A well-ordering on a set S is a relation \leq such that the following properties hold:

- (antireflexive:) $a \leq b$ and $b \leq a$ implies that $a = b$.
- (total:) $a \leq b$ or $b \leq a$, for any $a, b \in S$.
- (transitive:) $a \leq b, b \leq c$ implies that $a \leq c$.
- (least-element:) Every nonempty subset of S has a least element.

³Measure is a precise notion of the idea of 'length' extended to sets other than intervals. We want any good notion of measure to be translation-invariant (i.e. shifting an interval should not change its length), nontrivial (i.e. the length of $[0, 1]$ should be 1), and additive (i.e. the length of the union of two disjoint objects should be the sum of their lengths.)

- $V(G) = \mathbb{R}$,
- $E(G) = \{(s, t) : s - t - \sqrt{2} \in \mathbb{Q}\}$.

This graph has a chromatic number using the *ZFC* axiom system: call it $\chi^{ZFC}(G)$. It has some other chromatic number under the *ZFS* axiom system: call it $\chi^{ZFS}(G)$. The following two theorems show that these two chromatic numbers are remarkably distinct:

Theorem 2 $\chi^{ZFC}(G) = 2$.

Proof. Let

$$S = \{q + n\sqrt{2} \mid q \in \mathbb{Q}, n \in \mathbb{Z}\}.$$

Define an equivalence relation \sim on \mathbb{R} as follows:

$$x \sim y \text{ if and only if } x - y \in S.$$

Under the relation \sim , the elements of \mathbb{R} are broken up into various sets of elements that are all “equivalent” to each other under the relation \sim . Let $\{E_i\}$ be the collection of all of these equivalence classes.

Using the axiom of choice, pick one element y_i from each set E_i , and collect all of these elements in a single set E . Define the function $f : \mathbb{R} \rightarrow \mathbb{R}$ as follows:

$$f(x) = \text{the unique element } y_i \text{ in } E \text{ such that } x \sim y_i.$$

Now, define a two-coloring of \mathbb{R} as follows: for any $x \in \mathbb{R}$,

- color x red if and only if there is an odd integer n such that

$$x - f(x) - n\sqrt{2} \in \mathbb{Q}, \text{ and}$$

- color x blue if and only if there is an even integer n such that

$$x - f(x) - n\sqrt{2} \in \mathbb{Q}.$$

Why does this assign a unique color to every vertex? Well, we know that $x \sim f(x)$; so $x - f(x)$ is always of the form $q + n\sqrt{2}$, by construction. n is always either even or odd: so the above two cases assigns every vertex a unique color.

Furthermore, we claim that this coloring is a **proper** coloring: i.e. that no edge has two endpoints of the same color. To see why, take any edge $\{x, y\}$. By definition, we have to have $x - y = q + \sqrt{2}$, for some q . This means that $x \sim y$: i.e. that $f(x) = f(y)$! Consequently, we can rewrite

$$x - y = q + \sqrt{2}$$

as

$$\Rightarrow (x - f(x)) + (y - f(y)) = q + \sqrt{2}.$$

So: we know that there is some pair of integers m, n such that $x - f(x) - n\sqrt{2}$ and $y - f(y) - \sqrt{2}$ are both in \mathbb{Q} . By looking at the above equation, we can furthermore see that because there is exactly one copy of $\sqrt{2}$ on the right hand side above, these integers must have different parities! But this is just the statement that x and y have different colors under our coloring.

Therefore, this is a proper 2-coloring, and the chromatic number of G under ZFC is 2.

Without the axiom of choice, however, it turns out that we need a few more colors to color G :

Theorem 3 For G as above, $\chi^{ZFS}(G) > |\mathbb{N}| = \aleph_0$.

Proof. Consider the following massive hammer from analysis:

Theorem 4 (*Lebesgue Density Theorem*) If a set A has nonzero measure, then there is an interval I such that

$$\frac{\mu(A \cap I)}{\mu(I)} \geq 1 - \epsilon,$$

for any $\epsilon > 0$.

A proof of this theorem is very far beyond the scope of this class. A rough idea, however, is the following: given any set, we can approximate its measure by coming up with a collection of intervals that cover the set, and using this as an upper bound. One theorem you can prove is that if a set has measure d , you can come up with some collection of intervals that cover that set with measure only slightly more than d . Therefore, if you take one of those intervals, you'd expect the ratio of the "measure" of the set elements in that interval to the length of that interval to be very close to 1.

We mention it because it helps us prove the following lemma:

Lemma 5 If $A \subset [0, 1]$ and A doesn't contain a pair of adjacent vertices in G , then A has measure⁴ 0.

Proof. So: choose any set A of measure > 0 , and pick I such that

$$\frac{\mu(A \cap I)}{\mu(I)} \geq 99/100,$$

⁴The measure of a set S is defined as the infimum of the sum $\sum_{i=1}^{\infty} (a_i, b_i)$, where we range over all countable collections of intervals $\{(a_i, b_i)\}$ such that $\bigcup (a_i, b_i) \supset S$. We denote this number by writing $\mu(S)$.

for instance. Then, pick $q \in \mathbb{Q}$ such that $\sqrt{2} < q < \sqrt{2} + \mu(I)/100$, and define $B = \{x - q + \sqrt{2} : x \in A\}$. Then B has been translated by at most $1/100$ -th of the length of I : so we have that

$$\frac{\mu(B \cap I)}{\mu(I)} \geq 98/100.$$

So, because $(A \cap I) \cup (B \cap I) \subset I$, and both of these sets are almost all of I , we know that they must overlap! In other words, there's an element y in both A and B . However, if $y \in B$, we know that there is some $x \in A, q \in \mathbb{Q}$ such that $y = x - q + \sqrt{2}$.

This statement is precisely the claim that there's an edge between x and y ! Therefore, A contains an edge, as claimed.

With this lemma, our proof is pretty straightforward. Suppose that we could color \mathbb{R} with \aleph_0 -many colors, and that the collection of colors used is given by the collection $\{A_i\}_{i=1}^{\infty}$. Let $B_i = A_i \cap [0, 1]$; then we have that all of the B_i are disjoint and $\bigcup B_i = [0, 1]$. Consequently, we have that $\sum_{i=1}^{\infty} \mu(B_i) = \mu([0, 1]) = 1$; so at least one of the B_i 's have to have nonzero measure! This contradicts our above lemma; consequently, no such \aleph_0 -coloring can exist.