Algebraic GT | Instructor: Padraic Bartlett

# Homework 1: Cayley Graphs

*Week 5* | *Mathcamp 2014*

1. Draw the Cayley graph for the quaternion group $\langle a, b \mid a^2 = b^2 = (ab)^2 \rangle$.
2. Show that the dihedral group $D_{2n}$ discussed in class can be expressed via the presentation $\langle a, b \mid a^n = b^2 = (ab)^2 = 1 \rangle$. Find its Cayley graph.
3. For any odd $n$, find a group $G$ with generating set $S$ such that its Cayley graph is an oriented $K_n$. (An **oriented** $K_n$ is just a copy of the complete graph $K_n$ where we assign a direction to each edge. These graphs are also called **tournaments**.)
4. Let $Q_n$ denote the graph corresponding to the $n$-dimensional unit cube. Find a group $G$ with generating set $S$ such that its Cayley graph is the unoriented graph $Q_n$. (By an **unoriented** graph, we are asking that whenever we have an edge $(x, y)$ in our Cayley graph, we want to also have the reverse edge $(y, x)$.)
5. In class, we claimed that any Cayley graph must be vertex-transitive[1]. Prove that any Cayley graph is a vertex-transitive graph.
6. In class, we claimed that the converse of 5 fails: specifically, that the Petersen graph is vertex-transitive, and that no group/generating set pair can generate the Petersen graph as its Cayley graph. Prove this claim.
7. What platonic solids, thought of as undirected graphs, can be realized as Cayley graphs?
8. The following facts about cosets of a subgroup $H$ of a group $G$ are true:
   (a) For any $s \in G$, the right coset $Hs$ is equal to $H$ if and only if $s \in H$.
   (b) Two cosets $Hs, Ht$ are either completely identical or completely disjoint.
   (c) If $K$ is a coset and we form the set $Ks = \{k \cdot s \mid k \in K\}$, this set is also a coset.
   (d) The various possible cosets of $H$ **partition** $G$ into a collection of disjoint subsets. (In particular, this proves that the number of elements in $H$ must divide the number of elements in $G$.)
   (e) If $K$ is a coset of $H$ and $k$ is any element in $K$, then $Hk = K$.

   Do whatever you need to do to persuade yourself that these facts hold!

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[1]

**Definition.** Given two graphs $G_1, G_2$ with vertex sets $V_1, V_2$ and edge sets $E_1, E_2$, we say that a function $f : V_1 \to V_2$ is an **isomorphism** if the following two properties hold: (1) $f$ is a bijection, and(2) $(x, y)$ is an edge in $E_1$ if and only if $(f(x), f(y))$ is an edge in $E_2$. An **automorphism** on a graph $G$ is an isomorphism from that graph to itself.

Using this definition, we say that a graph $G$ is **vertex-transitive** if given any two vertices $v_1, v_2$ of $G$, there is an automorphism $f$ on $G$ such that $f(v_1) = v_2$. In essence, vertex-transitive graphs have a lot of symmetry: up to the labeling, we cannot distinguish any two vertices.