Electrical Networks and Graphs — Professor: Padraic Bartlett

Homework + Lecture 1: Electrical Networks and Random Walks

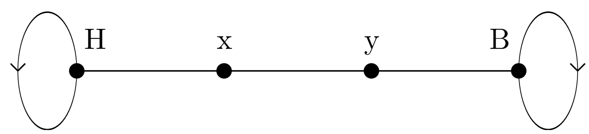
Week 2 — *Mathcamp 2014*

(Relevant source material: Doyle and Snell's "Random walks and electrical networks," which is available online here! Also, sections of Bollobas's text on Modern Graph Theory, various articles I've read, and probably other random things.)

Consider the following problem:

Problem. (The Lost Hedgehog's Walk.) Oh nyo! A hedgehog has gotten lost in the fog. Will it ever come home?

Specifically: consider the following model for a lost hedgehog's very simplified map of the universe:



There are in this world four possible locations: H, the hedgehog's camp, B, an all-devouring black hole that absorbs everything that accidentally wanders into it, and two intermediate locations x and y. Lost hedgehogs, left to its own devices, will randomly wander between these locations. Specifically: if it is at some vertex that is neither H nor B at time t, at time $t + 1$ it will choose via coinflip one of the neighboring vertices to its current location and wander there. If the hedgehog ever makes it home (i.e. wanders to H,) it is safe and is merrily reunited with its family. If it wanders to B, it is sucked into the black hole and never will be seen again.

Suppose the lost hedgehog starts at x. What are the hedgehog's chances of making it home? How can we model these kinds of behaviors?

1 Random Walks

For a model as simple as this one, it's remarkably simple to determine what happens! Specifically, let's consider the hedgehog's chances of making it home starting from **any** vertex v, not just x: for notational convenience, denote this probability as $p(v)$. What do we know about these values?

- $p(H) = 1$: if the hedgehog starts at home, it's happy and safe!
- $p(B) = 0$: if we've accidentally left the hedgehog inside of the black hole, we're not going to see it anytime soon.
- For $v \neq H, B$, we have $p(x) = \frac{1}{2}p(H) + \frac{1}{2}p(y)$, and $p(y) = \frac{1}{2}p(x) + \frac{1}{2}p(B)$. This is because a hedgehog at any vertex that's neither home or the black hole will choose between the two neighbors available to it with the same probability (1/2), and then travel to that respective vertex via that edge. So, its chances of survival are $\frac{1}{2}\cdot$ its chances at the vertex to its left, plus $\frac{1}{2}\cdot$ its chances at the vertex to its right.

This gives us the following four linear equations in four unknowns:

- $p(B) = 0$,
- $p(H) = 1$,
- $p(x) = \frac{p(H)+p(y)}{2}$,
- $p(y) = \frac{p(x)+p(B)}{2}$,

Solving this system tells you that $p(2) = \frac{1}{3}, p(3) = \frac{2}{3}$, and thus that our specific hedgehog at vertex x has a 2/3rds chance of making it home.

Let's consider a trickier version of the above problem. Suppose that instead of just a four-vertex path, we have some graph G that we want to model a hedgehog's walk on, with selected vertices H and B that denote the hedgehog's home / a point of no return, respectively; this lets us model things like city blocks. Also, let's attach weights w_{xy} to every edge in our graph, that denote the likelihood that our hedgehog will pick that edge over the other edges available to it; this lets us distinguish between things like clean, well-light main streets and sketchy alleyways.

Under this model, if we still let $p(x)$ denote the probability that from x we make it to H before reaching B, we have the following system:

- $p(H) = 1$.
- $p(B) = 0$.
- For $x \neq H, B$, we have

  $$p(x) = \sum_{y \in (\text{neighbors of } x)} p(y) \cdot \frac{w_{xy}}{w_x},$$

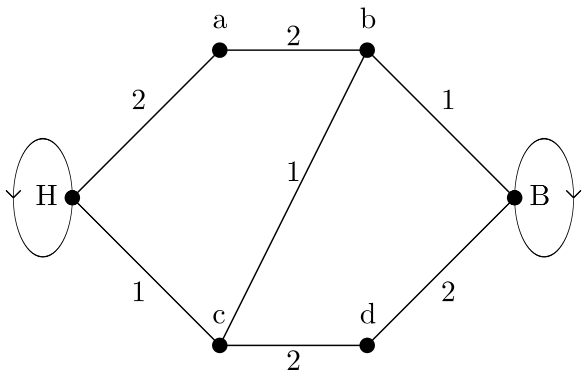
  where w_x is the sum of all of the weights of edges leaving x:

  $$w_x = \sum_{y \in (\text{neighbors of } x)} w_{xy}$$

  This is because a hedgehog at any vertex that's neither home or the black hole will choose between the neighbors available to it with probabilities weighted by the values w_{xy}: i.e. the probability that we travel to a neighbor y is just w_{xy}/w_x, the weight of the edge from x to y divided by the sum of the weights of all of the possible edges leaving x. Therefore, our probability $p(x)$ of making it to home before the black hole is just the **weighted average** over all of x's neighbors of the same event!

To illustrate this idea, we calculate a second example:

Problem. (The Hedgehog's Walk.) Consider the following second map for a hedgehog's walk:



There are in this world six possible locations: H, the hedgehog's home, B, a black hole, and four intermediate locations a, b, c, d, with weighted links between them as labeled. Suppose that a hedgehog starts off at one of these four locations. How likely are they to make it to the vertex H before the vertex B?

As noted above, we can turn this into a system of six linear equations in six unknowns:

- $p(B) = 0$,
- $p(H) = 1$,
- $p(a) = \frac{1}{2}p(H) + \frac{1}{2}p(b)$,
- $p(b) = \frac{1}{2}p(a) + \frac{1}{4}p(c) + \frac{1}{4}p(B)$,
- $p(c) = \frac{1}{4}p(H) + \frac{1}{4}p(a) + \frac{1}{2}p(d)$,
- $p(d) = \frac{1}{2}p(c) + \frac{1}{2}p(B)$.

Again, we can just solve these equations by your favorite method of dealing with systems of linear equations, to get

$$p(a) = \frac{12}{19}, p(b) = \frac{5}{19}, p(c) = \frac{8}{19}, p(d) = \frac{4}{19}.$$

Excellent! We have a general method for solving a problem. Let's put that aside for a second and consider a second problem that might seem unrelated at first:

2 Electrical Networks

We're going to talk about electrical circuits and networks here for a bit! If you've never ran into the concepts of voltage, current, conductance, or resistance before, that's OK. For our purposes, define these concepts as follows:

1. Voltage is just some function $v : V(G) \to \mathbb{R}^+$ that assigns a positive number $v(x)$ to each vertex x. In any circuit, we will have some vertex that is grounded; this vertex has $v(ground) = 0$. Similarly, we will declare that some source vertex has a potential difference of $1v$ from ground assigned to it: this vertex has $v(source) = 1$.

2. Current is just another function $i : E(G)^+ \to \mathbb{R}$ that assigns a number to each "oriented edge" $(x, y) \in E(G)^+$. We will usually denote the resistance of an edge as i_{xy}. We ask that $i_{xy} = -i_{yx}$, which is why we have the current pay attention to the orientation of edges: we want the flow of current in one direction on an edge to be $-1\cdot$ the flow of current in the opposite direction.

3. Resistance is a function $E(G) \to \mathbb{R}^+$ that assigns a positive number (measured in ohms, Ω) to each unoriented edge $\{x, y\} \in E(G)$. We usually denote the resistance of an edge as R_{xy}.

We ask that these functions preserve the following two properties:

- (Ohm's law:) The current across an edge $\{x, y\}$ in the direction (x, y), i_{xy}, satisfies

$$i_{xy} = \frac{v(x) - v(y)}{R_{xy}},$$

  where $v(x), v(y)$ are the voltages at x, y and R_{xy} is the resistance of the edge $\{x, y\}$.

- (Kirchoff's law:) The sum of the currents into and out of any vertex other than the grounded vertex or the "source" vertex is zero: i.e. for any vertex neither grounded nor hooked up to power, we have

$$\sum_{y \in N(x)} i_{xy} = 0.$$

For convenience's sake, we will also define the **conductance** of an edge $\{x, y\}$ as the reciprocal of its resistance: i.e. $C_{xy} = 1/R_{xy}$, and define the conductance of a vertex x as the sum of the conductances of the edges leaving it: i.e. $C_x = \sum_{y \in N(x)} C_{xy}$.

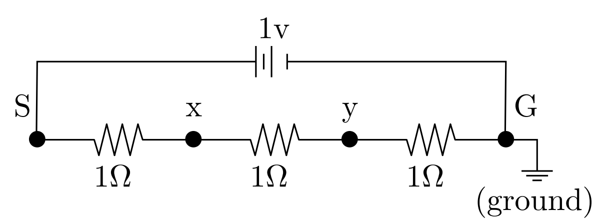
With these definitions made, the following problem is a fairly natural one to consider.

Problem. Suppose that we have an **electrical circuit**: i.e. a graph G with the following structure:

- The values R_{xy} have been defined for every edge.
- Some vertex G has been declared to be grounded, while another vertex S has been declared to be a "source" with a potential difference of $1v$ from ground.

Can we find $v(x)$ for every vertex in our graph?

We start by considering basically the first graph we studied in this lecture, P_4:



Specifically: we have taken the graph P_4 we studied in our first example of random walks, and turned it into a circuit as follows:

1. We replaced all of P_4's edges with resistors of unit resistance 1.
2. We grounded the vertex G, and created a potential difference of 1v across the vertices G and S.

The decorations on the graph above denote this transformation: i.e. attaching a vertex to ⏚ denotes that it is the ground vertex, ⊣||⊢ tells us that the vertex on the other side of this symbol from ground is a source vertex with a potential of some number of volts defined on it, −/\/\/− tells us that an edge is a resistor with the labeled resistance, etc.

In this setup, what happens? Well: we have that $v(G) = 0$, $v(S) = 1$, and for any vertex v not G or S,

$$\sum_{y \in N(v)} i_{vy} = 0;$$

i.e. for vertex x, we have

$$\begin{aligned} 0 = \sum_{y \in N(x)} i_{x,y} = i_{xS} + i_{xy} &= \frac{v(x) - v(S)}{R_{xS}} + \frac{v(x) - v(y)}{R_{xy}} \\ &= v(x) - v(S) + v(x) - v(y), \end{aligned}$$

which implies that $v(x) = \frac{v(S)+v(y)}{2}$; similarly, we can derive that $v(y) = \frac{v(x)+v(G)}{2}$. In other words, to find the voltages at the vertices x, y we're solving the same equations we did for our hedgehog's walk earlier: i.e. $v(x)$ is $2/3$, the probability that a hedgehog walking on our graph starting from x will make it to vertex S before vertex G!

3 Electrons Are Hedgehogs

Surprisingly, this property above – that our random walk and electrical network were, in some sense, the "same" – holds for all graphs! In the following lemmas, we make this idea concrete:

Lemma. Suppose that we have a connected graph G with edges weighted by some labeling w_{xy}. Define a **hedgehog's walk** starting at a vertex x in our graph as the following process:

- Initially, the hedgehog starts at x.
- Every minute, if a hedgehog is at some vertex z, it randomly chooses one of the elements $y \in N(z)$ with probability given by the weights on its edges– i.e. each neighbor has probability w_{zy}/w_z of being picked – and goes to that vertex.

Let a, b be a pair of distinguished vertices in our graph, and $p(x)$ be the probability that a hedgehog starting at the vertex x will make it to vertex b before vertex a.

Then $p(x) = v(x)$, if we turn our graph G into a electrical network with a connected to ground, a unit of electrical potential sent across a and b, and replace every edge $\{x, y\}$ of G with a resistor with conductance w_{xy}.

Proof. This is pretty much identical to what we just did. Specifically: we know from Ohm's law that

$$i_{xy} = \frac{v(x) - v(y)}{R_{xy}};$$

therefore, if we plug Ohm's law into Kirchoff's law, we have that whenever $x \neq a, b$, we have

$$\begin{aligned} &\sum_{y \in N(x)} \frac{v(x) - v(y)}{R_{xy}} = v(x) \cdot \left(\sum_{y \in N(x)} \frac{1}{R_{xy}} \right) - \sum_{y \in N(x)} \frac{v(y)}{R_{xy}} \\ \Rightarrow \quad & v(x) \cdot \left(\sum_{y \in N(x)} \frac{1}{R_{xy}} \right) = \sum_{y \in N(x)} \frac{v(y)}{R_{xy}} \\ \Rightarrow \quad & v(x) C_x = \sum_{y \in N(x)} C_{xy} v(y) \\ \Rightarrow \quad & v(x) = \sum_{y \in N(x)} \frac{C_{xy}}{C_x} v(y). \end{aligned}$$

But what is $\frac{C_{xy}}{C_x}$? It's the probability that a hedgehog starting at x chooses to travel to the vertex y, if we're picking neighbors of x with probabilities given by the C_{xy}'s! In this specific case, where all of our resistances are 1, this is just the chance that a hedgehog at vertex x will go to y in our random walk.

But this is the exact same equation we're asking $p(x)$ to satisfy: i.e. we want

$$p(x) = \sum_{y \in N(x)} (\text{chance hedgehog goes from x to y}) \cdot p(y) = \sum_{y \in N(x)} \frac{w_{xy}}{w_x} \cdot p(y).$$

The only other restrictions we have on our voltage or random walk is that $v(a) = p(a) = 0, v(b) = p(b) = 1$: in other words, the equations that we're asking our voltage function to satisfy are the same that we're asking our probability function to satisfy!

We have just shown that $p(x)$ and $v(x)$ are both solutions to the same sets of linear equations. To conclude that they are equal, then, we just have to show that there is a unique solution to these equations!

We do this by first making the following two observations:

Observation. Take any system of linear equations of the form obtained from these random walks on a connected graph[1]; i.e. a collection of equations of the form

$$p(x) = \sum_{y \in N(x)} \frac{w_{xy}}{w_x} \cdot p(y),$$

along with some boundary conditions $p(b_i) = c_i$. (In this sense, the "boundary" points are the values that we're given at the start of our system, while the rest of the points are the "interior" points whose values are determined by these weighted averages of their neighbors.)

Then the maximum and minimum values of $p(x)$ must occur on these boundary points.

[1]Finding a solution to this kind of a system is the process of solving a **Dirichlet problem**, if you want a formal name for reference in your reading.

Proof. This is a fairly easy proof. Suppose instead that x is a point in the interior of our graph, and that p attains its maximum at x. If we apply this observation to the equation

$$p(x) = \sum_{y \in N(x)} \frac{w_{xy}}{w_x} \cdot p(y),$$

we get

$$p(x) = \sum_{y \in N(x)} \frac{w_{xy}}{w_x} \cdot p(y) \leq \sum_{y \in N(x)} \frac{w_{xy}}{w_x} \cdot p(x) = p(x).$$

The equality of the far-left- and far-right-hand-sides forces the intermediate terms to be equal: i.e. we must have $p(y) = p(x)$, for every neighbor of x! Repeated applications of this argument will eventually give us that every vertex connected to x — i.e. every vertex in our graph, because our graph is connected — is equal to $p(x)$. In particular, this means that our boundary points have values equal to $p(x)$ as well.

An identical argument will show that having an interior point correspond to a minimum of $p(x)$ will force all of our vertices to be equal to that minimum as well. □

Observation. Again, take any system of linear equations of the form obtained from these random walks on a connected graph; i.e. a collection of interior equations of the form

$$p(x) = \sum_{y \in N(x)} \frac{w_{xy}}{w_x} \cdot p(y),$$

along with some boundary conditions $p(b_i) = c_i$.

Suppose that $p(x), q(x)$ are a pair of solutions to these equations. Then the mapping $r(x) = p(x) - q(x)$ is a solution to the same set of interior equations, where we replace all of the boundary conditions with the conditions $r(b_i) = 0$.

Proof. This is an even easier proof! Simply notice that if

$$p(x) = \sum_{y \in N(x)} \frac{w_{xy}}{w_x} \cdot p(y), \qquad q(x) = \sum_{y \in N(x)} \frac{w_{xy}}{w_x} \cdot q(y),$$

we have

$$r(x) = p(x) - q(x) = \sum_{y \in N(x)} \frac{w_{xy}}{w_x} \cdot (p(y) - q(y)) = \sum_{y \in N(x)} \frac{w_{xy}}{w_x} \cdot r(y).$$

Also, if $p(b_i) = c_i, q(b_i) = c_i$, then we have $r(b_i) = c_i - c_i = 0$. □

Given these two observations, we get the following corollary for free:

Corollary. If there is a solution to a system of linear equations of the form obtained from these random walks on a connected graph, it is unique.

Proof. Suppose we have two solutions $p(x), q(x)$ to such a system of linear equations. By our second observation, their difference $p(x) - q(x)$ is a solution to a system of linear equations where all of the boundary values are 0. By our first observation, the maximum and minimum of this $p(x) - q(x)$ is attained on the boundary. But this means that the maximum and minimum of $p(x) - q(x)$ is 0: i.e. that $p(x) = q(x)$! □

So, this tells us that a solution is unique if it exists. To finish our proof, we just need to simply note that a solution can exist! This is also not too hard, if we use a bit of linear algebra.

Take a system of linear equations of the form obtained from these random walks on a connected graph, where the boundary values are 0: i.e. a collection of interior equations of the form

$$p(x) = \sum_{y \in N(x)} \frac{w_{xy}}{w_x} \cdot p(y),$$

along with some boundary conditions $p(b_i) = 0$. By plugging in these boundary values into our interior equations, we can get a collection of n equations in n unknowns $p(x_1), \dots p(x_n)$ of the form

$$p(x_i) - \sum_{y \in N(x_i)} \frac{w_{x_i y}}{w_{x_i}} \cdot p(y) = 0.$$

In the standard fashion, turn these equations into a $n \times n$ matrix A by using the coefficients of these n equations as the entries in A's rows. Then a solution $p(x_i)$ for our set of equations corresponds precisely to a vector $\vec{p}$ such that $A\vec{p} = \vec{0}$.

On one hand, we know that a solution exists: simply set $\vec{p} = 0$. On the other hand, we know that any solution to our system is unique, as proven before! Therefore the only vector such that $A\vec{p} = \vec{0}$ is the all-zeroes vector: in other words, A is nonsingular! Therefore it has an inverse, A^{-1}.

Now, suppose that we were considering **any** boundary conditions $p(b_i) = c_i$. This would correspond to a collection of equations of the form

$$p(x_i) - \sum_{y \in N(x_i)} \frac{w_{x_i y}}{w_{x_i}} \cdot p(y) = d_i,$$

for coefficients d_i given by

$$d_i = \sum_{\text{boundary components } b_j} c_j \cdot \frac{w_{x_i b_j}}{w_{x_i}}$$

(where we assume $w_{x_i b_j} = 0$ if no edge connects x_i to b_j.)

Solutions to this system of equations correspond to vectors $\vec{p}$ that are solutions to the equation $A\vec{p} = \vec{d}$. Because A is invertible, such solutions exist! In particular, they are given by $\vec{p} = A^{-1}\vec{d}$. This proves that solutions exist and are unique, which was our claim! $\square$

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Homework 1

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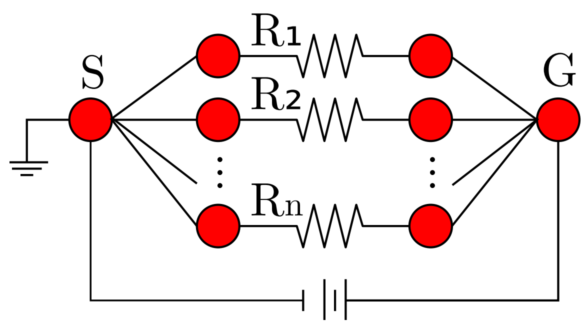
1. Prove the following claims about resistors, using Ohm's law and Kirchoff's law:

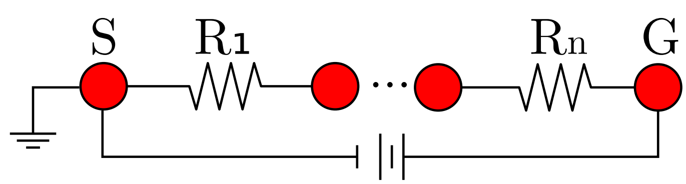
   (a) The effective resistance of the circuit below is the reciprocal of the sum of the reciprocals of the resistors in the circuit. In other words, the circuit

   has effective resistance given by the formula

   $$\frac{1}{R_{\text{eff}}} = \sum_{i=1}^{n} \frac{1}{R_i}.$$

   (b) The effective resistance of the circuit below is the sum of the resistors in the circuit. In other words,

   has effective resistance given by the formula

   $$R_{\text{eff}} = \sum_{i=1}^{n} R_i.$$

2. Suppose that we take the 2^n vertices of the n-dimensional cube, connect them all with resistors, ground the origin, and put a $1v$ potential difference between the origin and the point $(1, 1, \ldots 1)$. What is the resistance of this circuit?