Electrical Networks and Graphs	Professor: Padraic Bartlett

Homework + Lecture 3: Interpretations of Current

Week 2 — *Mathcamp 2014*

We started off our lecture here by expanding on the last lecture's discussion of what R_{eff} and i_{source} are for various circuits. After finishing this up, we moved on to the reasons we care about these quantities in this class; namely, the probabilistic interpretations of these objects!

1 Interpretations of Current

On the homework, we noted that the current across an edge (x, y) was proportional to (the expected number of paths from x to y minus the expected number of paths from y to x), up to scaling by the voltage we've established between the ground and the source. Does this idea still hold here? In other words: if A is our source vertex and G is our ground vertex, is there some connection between i_A and the total number of paths from A to G?

Well: calculating, we have

$$\begin{aligned} i_A &= \sum_{y \in N(A)} (v(A) - v(y)) \cdot C_{Ay} \\ &= \sum_{y \in N(A)} (v(A) - v(y)) \cdot \frac{C_{Ay}}{C_A} \cdot C_A \\ &= C_A \left(v(A) \sum_{y \in N(A)} \frac{C_{Ay}}{C_A} - \sum_{y \in N(A)} v(y) \frac{C_{Ay}}{C_A} \right) \\ &= C_A \left(v(A) - \sum_{y \in N(A)} v(y) \frac{C_{Ay}}{C_A} \right). \end{aligned}$$

What is this quantity? Well:

- We know that $v(A)$ is 1, by assumption.
- We also know that $\frac{C_{Ay}}{C_A}$ denotes the probability that a random walker will travel from the vertex A to the vertex y.
- Finally, we know that $v(y) = p(y)$, the probability that a walk starting at y will make it to A before G.

So: if we're starting at A and leaving to any of A's neighbors (which we pick with probability $\frac{C_{Ay}}{C_A}$), the chances of returning to A before making it to G is just $v(y)$. Therefore, the sum on the right inside of our parentheses is precisely the chances of starting at A and returning there before making it to G; consequently, 1 minus this sum is precisely the likelihood that we start at A and do not return there before wandering to G. Call this event, where we start at A and wander to G without returning to A, an "escape" event, and denote the probability of such an event happening p_{esc}.

If we plug this interpretation into our formula above, we get the following fantastically useful relation:

$$\frac{i_A}{C_A} = p_{\text{esc}}.$$

This lets us answer problems that otherwise would seem somewhat difficult to solve! Consider the following quick demonstration problem, derived from the first homework:

Problem. Take the three-dimensional unit cube, interpreted as a graph. Consider a random walker that starts at the point $(1,1,1)$, and wanders until it either returns to $(1,1,1)$ or $(0,0,0)$ — our source and sink vertices, respectively. What is the probability that our random walker starts at $(1,1,1)$ and returns to the origin?

Proof. In the language of the observation we just made, we are trying to find p_{esc} for our random walk. We start by turning this into a circuit as normal; i.e. ground the origin, set a potential of 1 volt at $(1,1,1)$, and replace each edge with a resistor with unit resistance. By our observation above, our problem's statement is equivalent to finding $i_{\text{source}}/C_{\text{source}}$!

This is not too hard. To do this, we make the following four very useful observations:

1. In our circuit-cube, the points $(1,0,0),(0,1,0)$, and $(0,0,1)$ are all at the same voltage. This is because (up to symmetry) these three points are **indistinguishable**! Informally, this is pretty intuitive: imagine taking our circuit and rotating it around the $x=y=z$ axis by $120°$. This doesn't physically change any of the connections/resistors/etc. in our circuit, so it should not change any of the properties of our circuit. But it permutes these three points! Therefore, the only difference between these three points in our circuit is just their "labeling" – physically speaking, they are all equivalent!

   Formally speaking, we can define this concept of "indistinguishable" as follows: a "circuit-property preserving" automorphism on a circuit with underlying graph G is any map from G to G that preserves all of the connections / source and sink / resistor values of our circuit. If there are any two points x, y with a circuit-property-preserving automorphism taking x to y, then x and y must have the same voltage — this is because our circuit cannot tell the difference between these two points, as it is possible to relabel the vertices in our circuit to exchange x and y!

   In practice, the informal definition here is more than sufficient for our purposes in this class.

2. The points $(1,1,0),(1,0,1),(0,1,1)$ are also all at the same voltage, by an argument completely identical to the one above.

3. Suppose that there is an edge in our graph $\{x,y\}$ with no current flowing along it. What happens when we change the resistance along this edge? In particular: does this change any of our circuit properties anywhere else in our graph?

   Well: Ohm's law says that $v(x) - v(y) = i_{xy}R_{xy} = 0$. In particular, this tells us both that

(a) the voltages at these two points must be equal, and

(b) changing the resistance between these two points does not effect the voltages of these two points!

Therefore, at x and y, changing the resistance doesn't effect the currents or voltages involving these two points. Consequently, if we "zoom out" to the rest of our graph, these properties are also unchanged; the only rules we have for determining $v(a), i_{ab}$ for vertices a, b were our boundary conditions $v(\text{source}) = 1, v(\text{ground}) = 0$ and our local averaging conditions $v(a) = \sum_{b \in N(a)} C_{ba} \cdot (v(b) - v(a))$.

So changing the resistance here doesn't change any of our graph properties elsewhere!

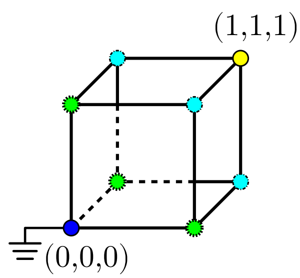
4. In particular, suppose that we have two vertices with the same voltage. Then there is no current flowing between those two vertices! Therefore, we can change the resistance between those two vertices to whatever we want.

   In particular, we can set it to 0. Physically, what does this mean? Well: we've effectively said that for an electron, there is "no difference" between x and y: i.e. there is no cost in any time/effort/etc to switch between x and y. In other words, we've **collapsed** the vertices x and y into one vertex!

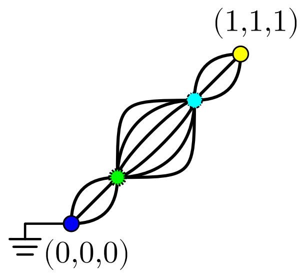
   Again: does this collapsing process change anything in our circuit? Well: we still satisfy Kirchoff's law at the new vertex xy, because the sum of currents through either x or y was zero, and we just added these objects together. Similarly, we still satisfy Ohm's law: because $v(x) = v(y) = v(xy)$, swapping in xy for x or y in any instance of Ohm's law doesn't change any of the equations being satisfied.

   Therefore, this collapse again doesn't change any of the voltages or currents throughout our circuit!

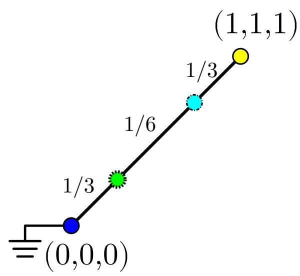
We can use these observations, along with the resistors in parallel/resistors in series observations from the HW and the second day's notes, to find R_{eff} for our cube. Namely: take our cube-circuit.



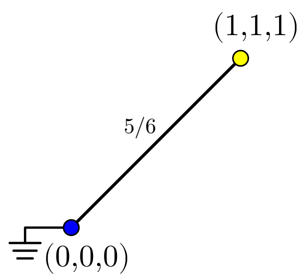
Use observations 1+4 to collapse the three one-1 points into a single vertex. Similarly, use observations 2+4 to collapse the three two-1's points into a single vertex.



Use our observations about resistors in parallel to collapse these multiple edges to single edges:



Use our observations about resistors in series to collapse these three edges into one edge:



Now, apply Ohm's law to deduce that $i_{\text{source}} = (v(\text{source}) - v(\text{sink})/R_{\text{eff}} = 6/5$, and therefore that $p_{\text{esc}} = \frac{6/5}{C_{\text{source}}} = \frac{6/5}{3} = 2/5$. (Recall that C_{source} is just the sum of the conductances of the edges leaving the source. In particular, because there are three edges leaving the origin all with resistance 1, this is just $\frac{1}{1} + \frac{1}{1} + \frac{1}{1} = 3$.)

So our probability of escape is 2/5!

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Homework 3

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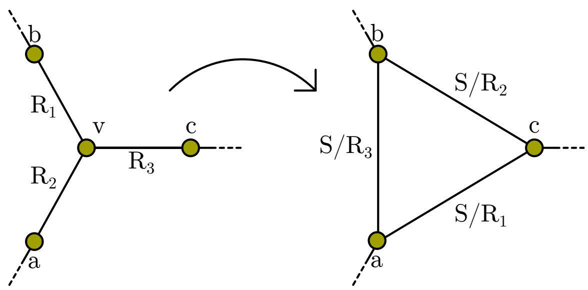
1. Take an arbitrary circuit. Define the **star-triangle** transformation as follows:



The star-triangle transformation. In the above diagram, $S = R_1R_2 + R_1R_3 + R_2R_3$.

This process takes a Y-configuration of resistors with resistances R_1, R_2, R_3 as labeled above, and replaces it with a Δ-configuration of resistors with resistances SR_1, SR_2, SR_3 as labeled above, with $S = R_1R_2 + R_1R_3 + R_2R_3$. Prove that this replacement does not change the overall resistance of the circuit in which it is performed.

2. In a graph G, a **delta-wye** exchange is a graph obtained from G by deleting the edges of a triangle v_1, v_2, v_3, adding a new vertex x, and creating new edges $v_1 \leftrightarrow x, v_2 \leftrightarrow x, v_3 \leftrightarrow x$. A **wye-delta** exchange is the reverse process. (Basically, this is the same process as described above, except without worrying about resistances.)

   Starting with the graph K_6, perform a sequence of these exchanges to get to the Petersen graph.

3. Combine problems 1 and 2 to find an alternate solution to problem 3 from Wednesday's HW!

   Namely: suppose you take the Petersen graph, and turn it into a circuit by replacing all edges with unit resistors, choosing any two nonadjacent vertices, and arbitrarily declaring one to be the source and the other to be ground. What is the effective resistance of this circuit?

   (Note: this is likely trickier than a more direct approach.)