Electrical Networks and Graphs — Professor: Padraic Bartlett

Homework + Lecture 4: Pólya's Random Walk Problem

Week 2 — *Mathcamp 2014*

Consider the following puzzle posed by Polya (amongst others):

Question 1. *Suppose that you have placed a random walker placed at the origin of a d-dimensional integer lattice $\mathbb{Z}^d$, and let it wander. Given enough time, will the random walker return to the origin? Or is there a nonzero chance that the random walker will wander forever without returning to the origin?*

Let's turn to $\mathbb{Z}^1$ as a quick warm-up. Our question, here, is whether a random walker starting at some point on the integer line (say the origin) will always return to the origin, or whether there's a nonzero chance that it wanders off forever.

However: all of our tools, as currently formulated, only apply to finite graphs! To study an infinite graph like $\mathbb{Z}^d$, then, we need to do the following:

- Let x be whichever node we're designating as the origin, and $G^{(r)}$ be the graph formed by taking all of the vertices connected to x by paths of length at most r.
- Turn this into a electrical network problem by soldering all of the vertices that are distance r from x together into one big ball (i.e. identifying all of these vertices together,) grounding them, putting one unit of voltage at x, and making all of the edges resistors with resistance 1. Then, via our earlier discussions, we can talk about the probability that a drunkard starting at x will make it to this point at distance r before returning to x. Denote this quantity as $p_{\text{esc}}^{(r)}$.
- Let p_{esc} be the limit $\lim_{r\to\infty} p_{\text{esc}}^{(r)}$. If this is nonzero, then there is some nonzero chance that our walker will wander forever; if this is zero, then our walker must eventually return to the origin.
- Notice that if it must eventually return to the origin, then it must eventually make it to any vertex w in G! This is because starting from the origin, we always have some nonzero chance to make it to w, and (because we return to the origin infinitely many times) we get infinitely many tries.

If G is a graph on which we return infinitely many times to the origin, we call G **recurrent**; if it is a graph where there is a chance that we will never return to the origin, we call G **transient**.

Theorem 2. *The one-dimensional lattice graph $\mathbb{Z}$ is recurrent.*

Proof. Let 0 be the origin, without any loss of generality. Using our earlier discussion, we know that

$$p_{\text{esc}}^{(r)} = \frac{i_0}{C_0} = \frac{1}{C_0}\cdot\frac{v(0)}{R_{\text{eff}}} = \frac{1}{C_0 R_{\text{eff}}}.$$

We know that the resistance of a string of r resistors in a row is r, from our earlier discussion about resistors in series. Consequently, because there are two such strings in parallel from the origin to distance r for any r, we know that their combined resistance is $\frac{1}{\frac{1}{r}+\frac{1}{r}} = \frac{r}{2}$. Therefore, because the conductance of the origin is $1 + 1 = 2$, we have

$$p_{\text{esc}}^{(r)} = \frac{1}{2 \cdot r/2} = \frac{1}{r}.$$

The limit as r goes to infinity of this quantity is 0; therefore, this walk is recurrent. $\square$

That wasn't so hard! However, dimensions greater than one pose greater difficulties. In particular, dealing with something like the resistance of an integer lattice is not something that our techniques are yet equipped to deal with. Symmetry arguments allow us to group some clusters of vertices together, but not as many as we like; as well, our series/parallel arguments are not very effective at dealing with vertices on a mesh!

As it turns out, these are not the only tools we have. Consider the following theorem of Rayleigh, that at first glance may seem too trivial to merit proving:

Theorem 3. *If any of the individual resistances in a circuit increase, then the overall effective resistance of the circuit can only increase or stay constant; conversely, if any of the individual resistances in a circuit decrease, the overall effective resistance of the circuit can only decrease or stay constant.*

In specific, cutting wires (setting certain resistances to infinity) only increases the effective resistance, while fusing vertices together (setting certain resistances to 0) only decreases the effective resistance.

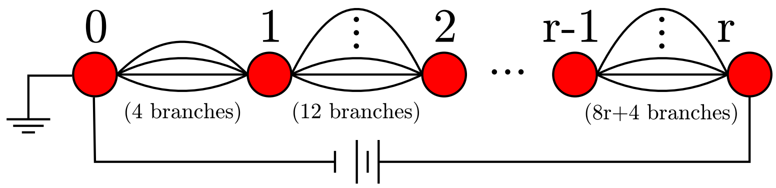
From a circuit perspective, this seems very trivial. If I replace a 1Ω resistor with a 10000Ω resistor, surely the overall resistance of my circuit has increased! However, from the random walk perspective (graph with source and sink, walker starts at source, wanders until it hits the sink or the source again) this is actually a very deep and surprising result. Effectively, we're claiming that no matter how you add edges to a graph, you can never increase the chance that a random walker encounters the source before the sink! Similarly, deleting edges can never make it less likely for our random walker to encounter the source before the sink.

For the rest of this talk, we're going to stick with the circuit perspective (and in particular not prove this result.) It's a beautiful but technically tricky thing to formally prove given the language of this class; if we have time on Saturday we'll return to it! For now, though, let's see what it can do for us:

Theorem 4. *The two-dimensional lattice graph $\mathbb{Z}^2$ is recurrent.*

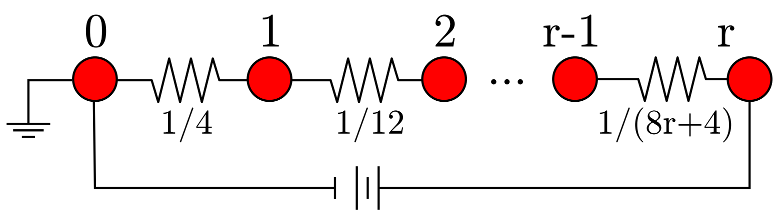
Proof. Take our graph, turn it into an electrical network with origin $= (0, 0)$, and perform the following really clever trick: for every r, let V_r be the collection of all of the vertices that are distance r from the origin under the taxicab metric (i.e. shortest length of a path.) Take our graph and **short** all of V_r's vertices into one huge clump, for each r: i.e. take the collection of all of the vertices at distance r, and just stick them all together! In essence, we are adding wires between all of the vertices at distance r with resistance 0, which (if you think of these wires not as connecting vertices that didn't use to be connected, but rather as replacing the wires of resistance "∞" between such vertices) is decreasing the resistance between certain vertices. We know that this reduces the overall resistance, because of Rayleigh's principle; therefore, we know that if this graph is recurrent, $\mathbb{Z}^2$ must be as well.

What does this process do to the graph $(\mathbb{Z}^2)^{(r)}$? Well, it produces the following picture:



Note that there are $8n+4$ edges between the vertices at distance n and the vertices at distance $n+1$ (a task we reserve for the homework!)

So: what is the resistance here? Well: if there are $8n+4$ resistors between node n and node $n+1$, we can regard our graph as equivalent to the path on $\{0, \dots r\}$ where the resistance between vertices n and $n+1$ is $\frac{1}{8n+4}$:



By adding these resistances together, we can finally calculate the effective resistance of this "shorted" $(\mathbb{Z}^2)^{(r)}$:

$$\sum_{i=1}^{r} \frac{1}{8i+4}.$$

This sum diverges to infinity! (If you haven't seen why this is this before, talk to me at TAU.) Therefore, the current on these graphs, and thus the $p_{\text{esc}}^{(r)}$'s,must converge to 0. So $(\mathbb{Z}^2)^{(r)}$ is also recurrent. $\square$

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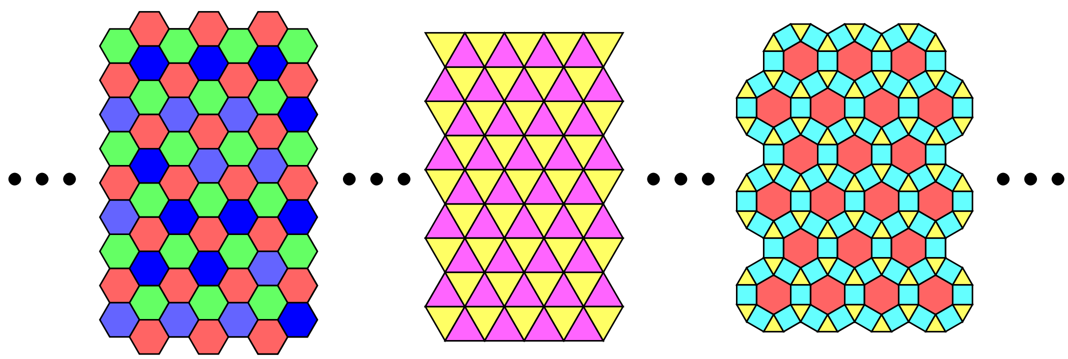
1. Prove the claim we made in class: that in $\mathbb{Z}^2$, the number of edges connecting points at distance n from the origin to points at distance $n+1$ is $8n+4$.

2. Suppose we are in $\mathbb{Z}^3$. How many edges are there between points at distance n and points at distance $n+1$ here?

3. Suppose that we have a random walker starting at some arbitrary point $\vec{a}$ in $\mathbb{Z}^3$, that at each time step picks one of the three possible axes and moves either one unit in the positive or negative direction with equal probability. Must our random walker eventually wander onto the plane $x+y+z=0$?

4. Suppose we have a random walker on the graph given by any of the 11 regular or semiregular tilings of the plane:



Some example tilings. More pictures and definitions at this Wikipedia link: http://en.wikipedia.org/wiki/Tiling_by_regular_polygons.

Will our random walker return to v with probability 1? Or is it possible that our random walker will wander forever on one of these tilings?