Mathcamp Crash Course	Instructor: Padraic Bartlett
Homework 1	
Week 1	*Mathcamp 2014*

Homework instructions: some of the problems below are labeled with the tag $(*)$. $(*)$ denotes that the problem in question is fairly fundamental to the topics we're studying, and that I want you to definitely solve. If you get stuck on any problem, or see a typo, find me! I can offer tons of hints and corrections.

This class is homework-required! What this means is that I'm expecting you to try every problem and to solve all of the $(*)$ ones. You are welcome (and encouraged!) to work with either each other during TAU or other times on the problem set; you can also bring questions to me or to other staff! We like talking about this stuff.

HW is due at the start of class every day! I'll try to look over solutions in between classes, and come up with comments for you in time for TAU. Relatedly: come find me at TAU each day to get your HW, and to talk about how you're doing in the class!

1. [$(*)$] Suppose that x is some variable. Form the following four mathematical statements:

   - A: the statement "$\frac{11}{3} > x > \frac{5}{3}$."
   - B: the statement "$x^2 = -1$."
   - C: the statement "$x^2 = 4$."
   - D: the statement "$x \neq 2$."

   Which of the following statements are true? Justify your answers.

   (a) $\forall x \in \mathbb{R}, C \Rightarrow A$ holds.
   (b) $\forall x < 0 \in \mathbb{R}, A \Rightarrow C$ holds.
   (c) $\exists x > 0 \in \mathbb{R}$ such that $C \wedge D$ holds.
   (d) $\exists x \in \mathbb{N}$ such that $C \wedge D \wedge A$ holds.
   (e) $\forall x \in \mathbb{Z}$, $D \Rightarrow (C \wedge D)$ holds.
   (f) $\forall x \in \mathbb{Z}, (A \vee B) \Rightarrow (\neg(C \vee D))$ holds.
   (g) $\neg(\exists x \in \mathbb{Q}$ such that $C \vee D$ holds.)
   (h) $\neg(\forall x \in \mathbb{Q}, \neg(A \wedge B \wedge C)$ holds.)

2. [$(*)$] Describe in words what the following "multiplicative inverses" property of the real numbers means:

   - **Inverses($\cdot$)**: $\forall a \neq 0 \in \mathbb{R}, \exists$ a unique $a^{-1} \in \mathbb{R}$ such that $a \cdot a^{-1} = 1$.

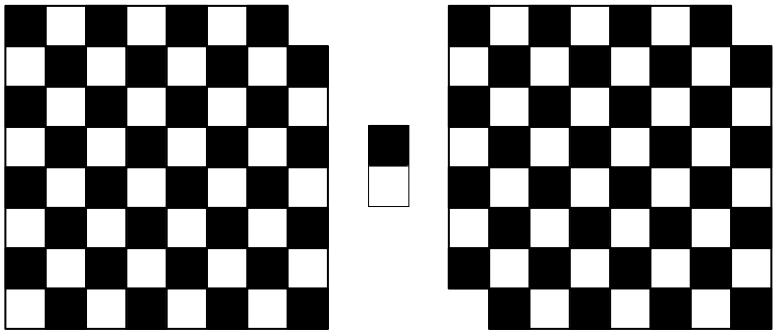
3. [$(*)$] Suppose that n is a natural number greater than 1. Look at $\langle \mathbb{Z}/n\mathbb{Z}, +, \cdot \rangle$. Show that this set satisfies the "multiplicative inverses" property above if and only if n is a prime number.

   In other words, show that n is a prime number if and only if the following property holds:

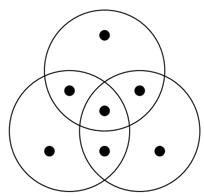
   - **Inverses($\cdot$)**: $\forall a \neq 0 \in \mathbb{Z}/n\mathbb{Z}, \exists$ a unique $a^{-1} \in \mathbb{Z}/n\mathbb{Z}$ such that $a \cdot a^{-1} \equiv 1$.

4. Prove that $\sqrt{2}$ is an irrational number.

5. Take a 8×8 checkerboard and punch out its top-right corner (drawn below.) Can you completely cover it with 2×1 rectangles that don't overlap and don't hang off the board? What if you remove its top right and bottom-left corner; can you cover it with 2×1 rectangles then?



6. Consider the following solitaire game:



The picture above contains three circles drawn in the plane. In each of the bounded regions formed by the intersections of these circles, we've placed a coin, which is white on one side and black on the other. All of the coins start with their black side up.

The moves you're allowed to perform in this game are the following:

- You can at any time flip all of the coins within any circle.
- Alternately, you can at any time take any circle and flip all of its white coins over to black.

Can you ever reach the following configuration? (Prove your claim.)