Mathcamp Crash Course	Instructor: Padraic Bartlett

Homework 4

Week 1 *Mathcamp 2014*

Homework instructions: some of the problems below are labeled with the tag $(*)$. $(*)$ denotes that the problem in question is fairly fundamental to the topics we're studying, and is something that you should probably be able to solve. If you get stuck on any problem, or see a typo, find me! I can offer tons of hints and corrections.

This class is homework-required! What this means is that I'm expecting you to read every problem, to solve all of the $(*)$ ones, and to solve at least some of the other problems.

HW is due at the start of class every day! I'll try to look over solutions in between classes, and come up with comments for you in time for TAU. Relatedly: come find me at TAU each day to get your HW, and to talk about how you're doing in the class!

1. [$(*)$] For any pair of real numbers a, b such that $a < b$, find a bijection between the sets $[0, 1]$ and $[a, b]$.

2. [$(*)$] Are the sets $\mathbb{N}$ and $\mathbb{N}^3$ (i.e. the set of all triples (a, b, c), where a, b, c are all natural numbers) the same size?

3. [$(*)$] Let A, B, C be a triple of sets, $f : A \to B$ an injective function, and $g : B \to C$ another injective function. Prove that $g \circ f$, the function that takes in any element $a \in A$ and outputs the result $g(f(a)) \in C$, is an injective function.

4. Can there ever be more words than numbers? Specifically: let's suppose that we're limiting ourselves to the 26-character Latin alphabet, and that the only kinds of things that can be **words** are finite strings of characters from the Latin alphabet. So things like

   - rabbit
   - barglearglesnarg
   - ssss
   - froyo

   are all possibly words. Call the set of all possible words $\mathbb{W}$. Is the set $\mathbb{W}$ the same cardinality as $\mathbb{N}$?

5. Define the **Cantor set** $\mathcal{C}$ as follows:

   - Start with the interval $[0, 1]$. Call this set C_0.
   - Remove the middle-third of this set, so that you have $[0, 1/3]$ and $[2/3, 1]$ left over. Call this set C_1.
   - Remove the middle-third of those two sets, so that you have $[0, 1/9], [2/9, 1/3], [2/3, 7/9], [8/9, 1]$ left over. Call this set C_2.
   - Repeat this process!

Define $\mathcal{C}$, the Cantor set, as the set made by taking all of the elements x such that x is in C_i, for every i.

(a) Find an element in $\mathcal{C}$.

(b) Show that $\mathcal{C}$ contains infinitely many elements.

(c) Can you make a bijection between $\mathcal{C}$ and $[0,1]$?

6. Find a bijection between the sets $[0,1]$ and $(0,1)$.

7. Let X denote the set made out of all possible sequences of natural numbers with finite length: i.e. for every element x of X, there is some length k such that x looks like some string $(n_0, n_1, \dots n_k)$, where the $n_1 \dots n_k$'s are all natural numbers. Is this set the same size as $\mathbb{N}$?

8. Let Y denote the set made out of all possible sequences of natural numbers with infinite length: i.e. for every element y of Y, y looks like some string $(n_0, n_1, \dots)$, where the elements n_i are natural numbers. Is this set the same size as $\mathbb{N}$?