Homework 4: Latin Squares and NP-Completeness

Homework Problems.

1. Prove the following claim made in class: the graphs $H_{n,m}$ are tripartite whenever n, m are multiples of 3.

2. Similarly, prove that whenever $H_{n,m}$ is tripartite, all three of its parts are the same size.

3. As well, prove that the graphs $H_{n,m}$ are **uniform** — in other words, given any vertex v in a part V_i of these tripartite graphs, the number of edges from this vertex to the $i+1$-th part of our graph is the same as the number of edges from this vertex to the $i-1$-th part of our graph. (I.e. $\deg_{i+1}(v) = \deg_{i-1}(v)$).

4. Finally, show that properties 1-3 are preserved under the gluings we defined in class, and thus hold for the graphs we created that correspond to instances of 3SAT.

5. Find the actual runtime, as a polynomial in n, of completing a partial Latin rectangle. Is it $O(n^2), O(n^3)$, or greater?

6. Prove Ryser's theorem:

   Theorem. (Ryser, 1951.) Suppose that P is a $n \times n$ partial latin square with the following properties:

   - There is a set of r rows named R, and a set of c columns named C, such that a cell (i, j) is not blank iff $i \in R$ and $j \in C$.
   - If $N(k)$ denotes the total number of times the symbol k is used in our entire square, then $N(k) \geq r + c - n$.

   Hint: this is basically like our Latin rectangles result, but weirder. Try expanding row-by-row to a Latin rectangle, and then using the fact that we know how to complete Latin rectangles!