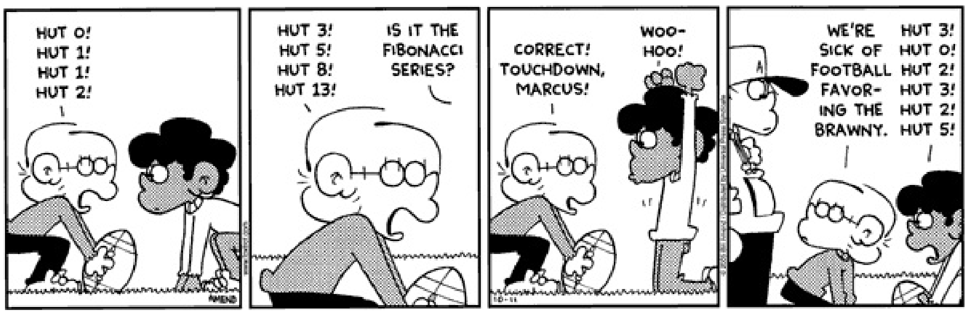
Math 116 | Professor: Padraic Bartlett

Homework 2: Generating Functions

Due Friday, Week 3 | *UCSB 2015*

Solve the following three problems!

1. Consider the Perrin sequence, which we can define recursively as follows: set $r_0 = 3, r_1 = 0, r_2 = 2$, and for all $n \geq 3$ define $r_n = r_{n-2} + r_{n-3}$.



The fourth panel of the FoxTrot cartoon above contains the first six values of the Perrin sequence.

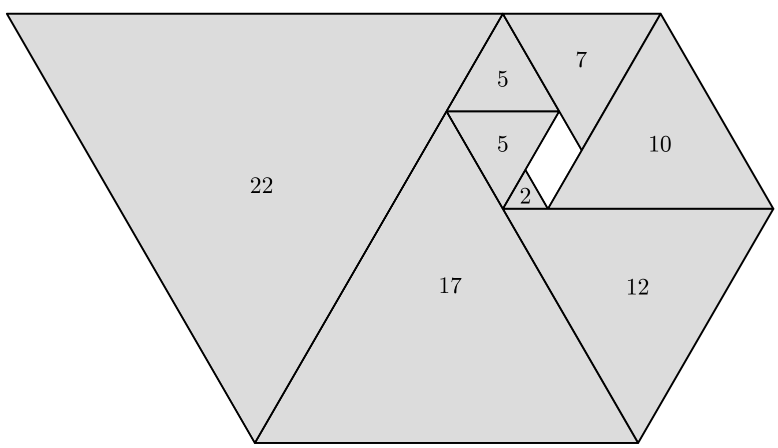
Let $R(x)$ denote the generating function $R(x) = \sum_{n=0}^{\infty} r_n x^n$ for the Perrin sequence.

(a) Prove that $R(x) = \frac{3-x^2}{1-x^2-x^3}$.

(b) Use this generating function to create a closed form for r_n.

(c) Prove that $r_n = r_{n-1} + r_{n-5}$, for all $n \geq 5$.

(d) Prove the following "spiral" relation for the Perrin numbers:



That is: start with the spiral drawn above, and rotating clockwise extend it by repeatedly drawing equilateral triangles with appropriate side lengths. Prove that the side lengths of these equilateral triangles are precisely the Perrin numbers starting from p_4.

2. Here's a fun/strange fact: if p is a prime number, then p divides r_p, the p-th Perrin number. We prove this in this problem, using the techniques of generating functions, as follows:

   (a) Define $Q(x) = R(x) - 3$; that is, $Q(x)$ is the same power series as $R(x) = \sum_{n=0}^{\infty} r_n x^n$, except we make the constant term zero for calculational convenience.
   Show that
   $$Q(x) = (-x) \cdot \frac{d}{dx}\left(\ln(1 - x^2 - x^3)\right).$$

   (b) Show that
   $$\left.\frac{d^p}{dx^p}\left(Q(x)\right)\right|_{x=0} = (-p) \cdot (p!) \cdot (\text{ the coefficient of } x^p \text{ in the Taylor series for } \ln(1 - x^2 - x^3)).$$

   (c) Show that on the other hand, if $Q(x)$ is any power series $\sum_{n=0}^{\infty} r_n x^n$, then
   $$\left.\frac{d^p}{dx^p}\left(Q(x)\right)\right|_{x=0} = p! \cdot r_p.$$

   (d) Using the above, prove that p is a factor of r_p whenever p is prime.

3. Let b_n denote the number of ways to create a $2 \times 2 \times n$ pillar using $2 \times 1 \times 1$ bricks.

   (a) Prove that the recurrence relation $b_n = 3b_{n-1} + 3b_{n-2} - b_{n-3}$ holds for all $n \geq 4$.

   (b) Find a closed form for the generating function $B(x) = \sum_{n=0}^{\infty} b_n x^n$.

   (c) Use this to find a closed form for the b_n's.

   (d) What is b_{25}?

Extra Credit

Solve the following problem if you're curious!

1. Let c_n denote the number of maximal independent[1] subsets of $\{1, 2, \ldots n\}$.

   (a) Prove that $c_n = r_n$ for any $n \geq 2$, where r_n is the Perrin sequence.

   (b) Consider the map $f : \{1, \ldots n\} \to \{1, \ldots n\}$ that cyclically shifts our set by $+1$; that is, $f(x) = x + 1$ for $x \neq n$ and $f(n) = 1$. For any subset A of $\{1, \ldots n\}$, let $f(A) = \{f(a) \mid a \in A\}$. Show that there is no independent set A such that $f(A) = A$. Use this fact to offer a non-generating-function proof that p divides r_p for any prime p.

[1]For this problem, a subset is called **independent** if it does not contain any two consecutive numbers, where we regard 1 and n as consecutive; it is called maximal if there is no set that contains this set as a proper subset and is also independent.