Math 116	Professor: Padraic Bartlett

Homework 3: More Generating Functions

Due Friday, Week 4 — *UCSB 2015*

In this HW set, there are **five** problems. Pick **three** of them to solve! If you solve more than three, only your **first three problems** will be graded. Choose wisely.

1. For any n, let o_n denote the number of ways to write n as an unordered sum of **odd** natural numbers: for example, $o_7 = 5$, because

$$1+1+1+1+1+1+1, 1+1+1+1+3, 1+3+3, 1+1+5, 7$$

are all of the ways to write 7 as a sum of odd natural numbers. Similarly, let d_n denote the number of ways to write n as an unordered sum of **distinct** natural numbers: for example, $d_7 = 5$, as

$$1+2+4, 1+6, 2+5, 3+4, 7$$

are all of the ways to write 7 as a sum of distinct natural numbers.

In the example above, $o_7 = d_7 = 5$. Prove that this wasn't a coincidence: that is, show that $o_n = d_n$ for every $n \geq 1$.

2. This problem consists of a very strange way to answer the following problem: in the expression $(\sqrt{2}+\sqrt{3})^{2014}$, can you determine the first few digits after the decimal place without using a calculator?

   (a) First, show that for any n, there are integers a_n, b_n such that $(\sqrt{2}+\sqrt{3})^{2n} = a_n + b_n\sqrt{6}$.

   (b) Now, show that the sequences a_n, b_n satisfy the recurrence relations

$$a_n = 5a_{n-1} + 12b_{n-1},$$
$$b_n = 2a_{n-1} + 5b_{n-1}.$$

   (c) If $A(x)$ denotes the generating function for the $\{a_n\}_{n=0}^{\infty}$ sequence, find a closed form for $A(x)$. Use this to prove that a_n has the closed form

$$a_n = \frac{1}{2}\left((5+2\sqrt{6})^n + (5-2\sqrt{6})^n\right).$$

   (d) Using the above plus our knowledge that $a_n + b_n\sqrt{6} = (\sqrt{2}+\sqrt{3})^{2n} = (5+2\sqrt{6})^n$, prove that

$$a_n = b_n\sqrt{6} + (5-2\sqrt{6})^n.$$

   Use this fact to find the first (say) three digits after the decimal place for $(\sqrt{2}+\sqrt{3})^{2014}$ without using a calculator.

3. The **Stirling numbers of the second kind** are defined as follows: for any natural numbers n, k, let $\left\{ n \atop k \right\}$ denote the number of ways to write the set $\{1, 2, \dots n\}$ as the union of k disjoint nonempty subsets. For instance, $\left\{ 3 \atop 2 \right\} = 3$, as there are precisely three ways to write $\{1, 2, 3\}$ as a union of two disjoint nonempty sets: $\{1, 2\} \cup \{3\}, \{1, 3\} \cup \{2\}, \{2, 3\} \cup \{1\}$.

   We can use the sieve method to find a very nice formula for these numbers, as follows. Take n labeled balls and consider the process of placing these balls into k labeled boxes. Let Ω denote the k^n many ways of placing these balls into boxes. Finally, for any $1 \leq i \leq k$, let p_i denote the property that box i is empty, and let $P = \{p_1, \dots p_k\}$ be the collection of all of these properties.

   (a) Prove that $k! \cdot \left\{ n \atop k \right\}$ is just the number of elements of Ω that satisfy no properties.
   (b) If S is any subset of our properties and $A(S)$ denotes the number of elements of Ω that satisfy all of the properties in S (and maybe more), show that $A(S) = (k - |S|)^n$.
   (c) In class, we defined $n_r = \sum_{S \subseteq P : |S| = r} A(S)$. Use (b) to find a formula for n_r in terms of k, n, r
   (d) Use the $N(x-1) = E(x)$ trick to prove the following formula for the Stirling numbers:

$$\sum_{r=0}^{\infty} \binom{k}{r} (k-r)^n (-1)^r = k! \cdot \left\{ n \atop k \right\}.$$

4. In class, we proved that if $A(x) = \sum_{n=0}^{\infty} a_n x^n$ was a generating function such that $A(x) = \frac{1}{(1-x)^2}$, then $a_n = n + 1$ for every n.

   On the third quiz, we looked at a generating function $B(x) = \sum_{n=0}^{\infty} b_n x^n$ such that $B(x) = \frac{1}{(1-x)^3}$. If you successfully completed this quiz problem, then you likely proved that $b_n = \frac{(n+1)(n+2)}{2} = \binom{n+2}{2}$ for all n.

   Generalize this as follows: for any natural number k, let $C(x) = \sum_{n=0}^{\infty} c_n x^n$ be a generating function such that $C(x) = \frac{1}{(1-x)^k}$. Prove that $c_n = \binom{n+k-1}{k-1}$ for all n.

5. We close with one last example of the sieve method. For any natural number n, let Ω denote the collection of all subsets of size n from $\{1, 2, \dots 2n\}$. For any $1 \leq i \leq n$, let p_i denote the property that one of these given subsets does **not** contain the number i, and let $P = \{p_1, \dots p_n\}$ denote the collection of these n properties. (Notice that our properties are only concerned with whether our subsets contain elements from $\{1, \dots n\}$, and don't care about the other $\{n+1, \dots 2n\}$ elements.)

   (a) Show that $A(S)$ is $\binom{2n-|S|}{n}$.
   (b) Show that $n_r = \binom{n}{r}\binom{2n-r}{n}$.
   (c) Finally, use the sieve method to show that for any n, we have the identity

$$1 = \sum_{r=0}^{\infty} \binom{n}{r} \binom{2n-r}{n} (-1)^r.$$