# Math 104A - Final Projects

**Due 8/1 at 11:59PM**

**Instructions**: Choose one of the following projects from physics/finance and write a report which answers each of the stated questions. Give your numerical results either in a plot (as indicated) or in a well-organized table. All codes must be written by you **and only by you** using the algorithms discussed in lecture and homeworks. You may consult the book and use the internet, but you are **not allowed to work with other students**.

Turn in either to me, my email's inbox, or my mailbox in SH 6623 by **August 1, 11:59PM**. Good luck!

# Project 1: Diffraction Limited Imaging

Diffraction manifests itself in the apparent bending of waves around small obstacles and the spreading out of waves past small openings. Not only does this apply to sound waves and waves in water, but it also applies to light. In optical imaging systems such as microscopes, telescopes, and cameras, there is a fundamental maximum to the resolution, which is due to diffraction. In this project, you will explore the basics of diffraction through single-slit experiments and circular apertures, and apply your results to obtain the angular and spatial resolution of various imaging systems.

**1(a)** Consider a plane with an infinite slit of very small width. One side of the plane is illuminated with monochromatic (i.e. all the same wavelength) light, and diffracts through the thin slit. If the slit is wider than one wavelength, then there will be interference effects on the other side of the plane.

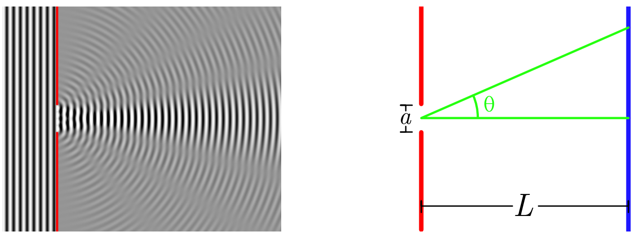



Figure 1. Light approaches the slit in the plane (in red) from the left. Once the light diffracts through the slit, interference causes the intensity of the light to be distributed unequally in space.

Suppose that you put an imaging plate some distance $L$ beyond the plane of the slit, the width of the slit is $a$, and the wavelength of the light is $\lambda$. Assuming that $\frac{a^2}{L\lambda} \ll 1$, the intensity profile is approximated by

$$I(\theta) = I_0 \left[ \frac{\sin\left(\frac{a\pi\sin(\theta)}{\lambda}\right)}{\frac{a\pi\sin(\theta)}{\lambda}} \right]^2, \tag{1}$$

where $I(\theta)$ is the intensity at a given angle $\theta$ and $I_0$ is the original intensity of the light. Give a plot of the intensity profile using the values $\lambda = 450$ nm, $I_0 = 1$, and $a = 10\lambda, 5\lambda, 2\lambda, \lambda$. Be sure to label your plot clearly. What do you notice about the intensity profile as $a \to \lambda$?

**1(b)** Write a code that finds the maxima of $I(\theta)$ with $\lambda = 450$ nm, $I_0 = 1$ and $a = 10\lambda$. You may use any of the root finding methods discussed in class or the homework. Be sure to justify your answers.

**2(a)** Now consider light incident on a plane with a very small circular aperture. The diffraction of the light results in what is called an "airy disk":

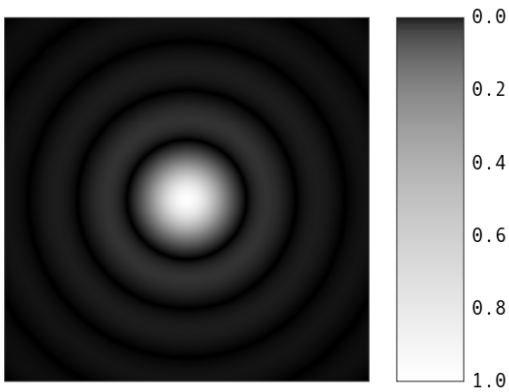



Figure 2. Diffraction pattern from a circular aperture, also known as an "airy disk." Note the first dark ring around the central light circle.

The intensity of light at a given angle $\theta$ is given by

$$I(\theta) = I_0 \left[\frac{2J_1(ka\sin(\theta))}{ka\sin(\theta)}\right]^2, \tag{2}$$

where $k = \frac{2\pi}{\lambda}$, $a$ is the radius of the aperture, and $J_1$ is a Bessel function of the first kind. In this case, $J_1$ is given by

$$J_1(x) = \frac{1}{\pi}\int_0^{\pi} \cos(t - x\sin(t))dt. \tag{3}$$

Write a code that implements one of the methods given in class to produce an approximation for $J_1(x)$, and use this approximation to give a plot for the intensity function $I(\theta)$. Use $\lambda = 450$ nm, $I_0 = 1$ and $a = 10$.

**2(b)** Using your approximation from part (a), interpolate $J_1(x)$ through the points $x_0 = 3.7, x_1 = 3.8$, and $x_2 = 3.9$. Use this interpolation to approximate the first positive root $\alpha$ of $J_1(x)$.

**2(c)** Imagine that there are now two circular apertures in the plane, each of which produce an "airy disk." The two images of the airy disks are said to be **resolved** when principal diffraction maximum of one disk (i.e. the center of the disk) coincides with the first minimum of the other (i.e. the first dark ring). Using the fact that the intensity of the first minimum of an airy disk is zero, we get that

$$I(\theta) = 0 \quad \Rightarrow \quad J_1(ka\sin(\theta)) = 0 \quad \Rightarrow \quad ka\sin(\theta) = \alpha.$$

The particular $\theta$ that satisfies this is called the **angular resolution**. The Apollo lunar landers are about 5 meters wide, at about 382,000 km away from the surface of the earth, which corresponds to 0.003 arcseconds ($1.45444104\times 10^{-8}$ radians). Assuming that the lunar lander gives off light of wavelength 570 nm, find the radius of the aperture (i.e. the radius of the telescope) to get an angular resolution of 0.003 arcseconds. Is there such a telescope on Earth?

The angular resolution is easily converted into the **spatial resolution** $\Delta l$

by multiplying the previous equation by the distance to the object. For a microscope, this distance is very close to the focal length $f$ of the objective. Then

$$ka \sin(\theta) = \alpha \quad \Rightarrow \quad \Delta l = \frac{f\alpha}{ka}.$$

Using this result, find the spatial resolution of a microscope with focal length $f = 5$ mm, aperture radius $a = 1$ mm, with a light source emitting light at wavelength $\lambda = 450$ nm. Would a human hair be resolvable in this microscope?

# Project 2: Double Pendulum, All The Way

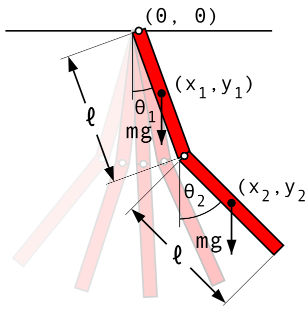
A double pendulum is a pendulum with another pendulum attached to its end. In this project, you will explore the motion of a double pendulum.

**1** The motion of a single pendulum is governed by the differential equation

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\sin(\theta) = 0 \tag{4}$$

Convert this second order differential equation into a system of first order differential equations. Write a code that implements the Runge-Kutta fourth order method for systems of equations to solve it. Solve for the interval $0 \le t \le 10$ with initial conditions $\theta(0) = \frac{\pi}{2}, \theta'(0) = 0$. Assume that $l = 1$. Give plots of $\theta$ vs. $t$, $\theta'$ vs. $t$, and $\theta'$ vs. $\theta$. Be sure to label your plots clearly.

**2(a)** A double pendulum consisting of two rods both of length $l$ and mass $m$ are hung from the ceiling. Let the angle that the first rode makes with the vertical be denoted by $\theta_1$, and the angle the second makes with the vertical by $\theta_2$.




The equations of motion governing the two rods is given by

$$\dot{\theta}_1 = \frac{6}{ml^2}\frac{2p_{\theta_1} - 3\cos(\theta_1 - \theta_2)p_{\theta_2}}{16 - 9\cos^2(\theta_1 - \theta_2)}, \tag{5}$$

$$\dot{\theta}_2 = \frac{6}{ml^2}\frac{8p_{\theta_2} - 3\cos(\theta_1 - \theta_2)p_{\theta_1}}{16 - 9\cos^2(\theta_1 - \theta_2)}, \tag{6}$$

$$\dot{p}_{\theta_1} = -\frac{ml^2}{2}\left[\dot{\theta}_1\dot{\theta}_2\sin(\theta_1 - \theta_2) + \frac{3g}{l}\sin(\theta_1)\right], \tag{7}$$

$$\dot{p}_{\theta_2} = -\frac{ml^2}{2}\left[-\dot{\theta}_1\dot{\theta}_2\sin(\theta_1 - \theta_2) + \frac{g}{l}\sin(\theta_2)\right], \tag{8}$$

where $p_{\theta_1}$ and $p_{\theta_2}$ are the angular momenta of the rods. Modify your code from part 1 to give an approximate solution to the system. Note that the differential equation is of the form

$$\dot{\mathbf{y}} = f(t, \mathbf{y}, \color{red}{\dot{\mathbf{y}}}).$$

You must determine how to handle the term in red. Use the interval $0 \le t \le 100$, with **two** sets of initial conditions

(1) $\theta_1(0) = \frac{\pi}{2}, \quad \theta_2(0) = \frac{\pi}{2}, \quad p_{\theta_1}(0) = 0, \quad p_{\theta_2}(0) = 0,$

(2) $\theta_1(0) = \frac{\pi}{2} + 0.001, \quad \theta_2(0) = \frac{\pi}{2} + 0.001, \quad p_{\theta_1}(0) = 0, \quad p_{\theta_2}(0) = 0.$

Assume that $l = 1$. Give plots for $\theta_1$ vs. $t$ and $\theta_2$ vs. $t$ for each set of initial conditions. Be sure to label your plots clearly.

**2(b)** The $(x, y)$-coordinates of the tip of the second rod are given by

$$x = l(\sin(\theta_1) + \sin(\theta_2)), \quad y = -l(\cos(\theta_1) + \cos(\theta_2)).$$

Using these transformations, plot the path of the tip of the second rod for both sets of initial conditions. One of the conditions for a dynamical system to be classified as **chaotic** is that the system must be sensitive to initial conditions. That is, two initial conditions which are very close to each other produce significantly different results. Do the two rods exhibit this property? Justify your answer.

# Project 3: Black-Scholes-Merton Option Pricing

When making investments in an asset in the marketplace (such as a stock), there are typically substantial risks in the future value of the asset. To facilitate management of these risks, banks sell contracts to protect investors against either large increases or large decreases in the value of an asset. A common class of contracts used for this purpose are referred to as options. For example, to manage price changes of a stock, a European Put Option is a contract which gives the holder the right to sell a given amount of the stock at a specified price $K$ (called the strike price) at a specific time in the future $T$ (called the maturity time). Similarly, a European Call Option gives the holder the right to buy at a specified price $K$ at a specific time $T$ in the future.

An important problem for banks who buy and sell such contracts is to determine a reasonable price for the contracts. Given the nature of the contracts, the price charged by a bank must somehow reflect the current value of the asset in the marketplace while at the same time reflecting the future liabilities the bank assumes by issuing the contract. This is in general a challenging problem given all of the uncertainties in the future behavior of the marketplace. However, when certain assumptions are made about the marketplace such a price can be determined. A well-known approach used in practice is the Black-Scholes-Merton Option Pricing Theory.

From the Black-Scholes-Merton Option Pricing Theory, the price of a European call option $c$ and a put option $p$ are given by:

$$c(s_0, K, T) = s_0 N(d_1) - K e^{-rT} N(d_2) \tag{9}$$

$$p(s_0, K, T) = K e^{-rT} N(-d_2) - s_0 N(-d_1), \tag{10}$$

where $s_0$ is the current price of the stock, $K$ is the agreed upon strike price, and $T$ is the maturity time in years. Also,

$$d_1 = \frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{s_0}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)T\right]$$

$$d_2 = \frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{s_0}{K}\right) + \left(r - \frac{1}{2}\sigma^2\right)T\right],$$

where $r$ is the current compounding interest rate and $\sigma^2$ is a parameter characterizing how much prices are expected to fluctuate in the marketplace. Finally, $N$ denotes the cumulative normal distribution function:

$$N(d) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d} e^{-\frac{y^2}{2}}\, dy.$$

In order to use formulas (9) and (10), the expressions above must be numerically evaluated. In this project, you will develop codes to price various call and put options.

**1(a)** Implement a code for a function which takes $d$ as input and returns $N(d)$. To approximate the $-\infty$ term, use a lower bound of at least -10. You may use any of the methods discussed in lecture and in the homeworks. Using your method, evaluate $N(0)$ to within $10^{-3}$ (note that $N(0) = \frac{1}{2}$).

**1(b)** Assuming that the current price of Amazon stock is \$300 today, compute the price of a put option which gives a holder the right to sell the stock 100 days from today for \$299. Assume that the current compounding interest rate $r = 0.05$ and the market volatility is $\sigma^2 = 0.2$.

**1(c)** Make a plot of the price of the put option in part (b) as the strike price $K$ is varied from 250 to 350. Give a labeled plot of $p$ vs. $K$. Discuss what happens to the price of the put option as the strike price is increased. Can you explain intuitively why this is expected to happen?

**2(a)** Assuming that the current price of Apple stock is \$300 today, compute the price of a call option which gives the holder the right to buy the stock 100 days from today for \$299. Assume that the current compounding interest rate $r = 0.05$ and the market volatility is $\sigma^2 = 0.2$.

**2(b)** Make a plot of the price of the call option in part (a) as the strike price $K$ is varied from 250 to 350. Give a labeled plot of $c$ vs. $K$. Discuss what happens to the price of the call option as the strike price is increased. Can you explain intuitively why this is expected to happen?

**3** Suppose an investor goes long (buys) a call option and goes short (sells) a put option. Using your data from parts 1(c) and 2(b), make a plot of $c - p$ vs. $K$ (now assuming they are for the same stock). The price of a forward contract, in which two parties agree to a selling price of an asset at some future time $T$ is given by $f(s_0, K, T) = s_0 - Ke^{rT}$. Add to your plot of $c - p$ a plot of the price of the forward contract. How do they compare? Can you explain why? The strike price at which the forward contract is worth 0 is called the "fair price" of the contract. At this strick price, neither party makes a net financial gain by entering into the contract. For what value of $K$ does the forward contract become worth zero for the contract with the parameters above.? This relationship between the forward contract and the portfolio of the Euorpean call and put option is call "put-call parity".

# Project 4: Markowitz Portfolio Theory

When making investments in the marketplace, a trade-off usually needs to be made between the expected return (payoff) of an asset and the riskiness in obtaining that return or a loss. The central tenet of Markowitz Portfolio Theory is that if two investment opportunities have the same expected return, then the one which is less risky is more desirable to investors. When faced with the opportunity to invest in many different assets an interesting problem arises in how to choose an optimal portfolio, which attempts to maximize the investment return while minimizing risks. In this project you will explore one model of investments which attempts to capture these trade-offs and use this model to construct optimal portfolios for a collection of assets.

Notation: from a collection of assets, we denote the expected return of the $i^{\text{th}}$ asset by $\mu_i$, and the riskiness of the $i^{\text{th}}$ asset by $\sigma_i^2$, the variance.

**1(a)** Consider two assets with expected returns $\mu_1, \mu_2$, variance $\sigma_1^2, \sigma_2^2$, and covariance $\sigma_{1,2}$. From Markozitz portfolio theory, a portfolio which invests a fraction $w_1$ of an investor's wealth in asset 1 and a fraction $w_2$ in asset 2 has the expected return

$$\mu_p = w_1\mu_1 + w_2\mu_2 \tag{11}$$

and variance

$$\sigma_p^2 = w_1^2\sigma_1^2 + 2w_1w_2\sigma_{1,2} + w_2^2\sigma_2^2. \tag{12}$$

Suppose an investor wishes to invest \$100,000 in the assets to get a return $\mu_p = 0.08$. Since there are only two assets, the weights are expressed as $w_1 = \alpha, w_2 = 1-\alpha$. Let the assets have

$$\mu_1 = 0.03, \quad \mu_2 = 0.09, \quad \sigma_1 = 0.2, \quad \sigma_2 = 0.4, \quad \sigma_{1,2} = -0.02.$$

Write a code which implements both the bisection method and Newton's method to find the value of $\alpha$ which makes $\mu_p = 0.08$ in equation (11). This can be solved by hand easily, so check that your code returns the correct result. What are the resulting weights $w_1$ and $w_2$? What is the variance $\sigma_p^2$ of the portfolio.

**1(b)** The fluctuations in the future value of the investment portfolio is modeled by the range $[V_1(t), V_2(t)]$, where

$$V_1(t) = we^{\mu_p t - \sigma_p\sqrt{t}}, \tag{13}$$

$$V_2(t) = we^{\mu_p t + \sigma_p\sqrt{t}}. \tag{14}$$

Give a plot for $V_1$ and $V_2$ for $t \in [0, 1]$ for the investment of $w = \$100,000$ made in part (a). Suppose you were given a choice between investing $w$ in the portfolio, or putting $w$ in a bank account paying a continuous compounding rate of 4%. Which choice would you take, and why? Justify your answer carefully and fully.

**1(c)** Suppose the investor wants most to reduce the riskiness of the investment made in the two assets. Use your code to determine the optimal value of $\alpha$

which minimizes the variance of the portfolio for *any* return. That is, find the zero of the function

$$\lambda_1(\alpha) = \frac{\partial \sigma_p^2}{\partial \alpha}.$$

Again, this is easily solved by hand, so check that your code gives the correct result. What are the weights $w_1$ and $w_2$? What is the return $\mu_p$ of this optimal portfolio? Give a plot for the fluctuation range $V_1$ and $V_2$ for $t \in [0, 1]$. Would you make this investment over putting $w$ in the bank account? Why?

**2(a)** Now consider certain assets (e.g. a factory) which has an economy of scale, so that the expected return may in fact increase as more resources are invested in the asset. Consider assets with

$$\mu_1(w_1) = 0.0005e^{3w_1}, \quad \mu(w_2) = 0.07,$$

$$\sigma_1^2(w_1) = e^{-3w_1}, \quad \sigma_2^2(w_2) = 0.4, \quad \sigma_{1,2} = -0.01.$$

With $w_1 = \alpha$ and $w_2 = 1 - \alpha$, use your code to find $\alpha$ which gives a portfolio with expected return $\mu_p = 0.05$. Allow $\alpha$ to vary in the range $[-1, 1]$, where negative weights correspond to going short on an asset. What are the resulting weights $w_1$ and $w_2$? What is the variance $\sigma_p^2$ of this portfolio? Assuming the investor wishes to invest a cool $w = \$1,000,000$, give a plot of $V_1(t)$ and $V_2(t)$ for $t \in [0, 1]$. Would you make this investment over putting $w$ in the bank account? Why?

**2(b)** Suppose the investor wishes to invest in the least risky portfolio comprised of the two assets in part (a). Find the value of $\alpha$ which gives this portfolio. Allow $\alpha$ to vary in the range $[-1, 1]$, where negative weights correspond to going short on an asset. What are the resulting weights $w_1$ and $w_2$? What is the expected return of this portfolio $\mu_p$? What is the variance $\sigma_p^2$ of this portfolio? Give a plot of $V_1(t)$ and $V_2(t)$ for $t \in [0, 1]$, using the same $w$ as in part (a). Would you make this investment over putting $w$ in the bank account? Why?