Math 104A - Homework 1

Section 1.1 - 2, 4a, 4c, 8, 12a, 12b, 24

2 Find intervals containing solutions to the following equations.

(a) $f(x) = x - 3^{-x} = 0$.

Since $f(0) = -1$ and $f(1) = \frac{2}{3}$, by the intermediate value theorem, there is some solution of f in the interval $[0, 1]$.

(b) $f(x) = 4x^2 - e^x = 0$.

Since $f(0) = -1$ and $f(1) = 4 - e$, by the intermediate value theorem, there is some solution of f in the interval $[0, 1]$.

(c) $f(x) = x^3 - 2x^2 - 4x + 2 = 0$.

Since $f(0) = 2$ and $f(1) = -3$, by the intermediate value theorem, there is some solution of f in the interval $[0, 1]$.

(d) $f(x) = x^3 + 4.001x^2 + 4.002x + 1.101 = 0$.

Since $f(-3) = -1.896$ and $f(-2) = 1.101$, by the intermediate value theorem, there is some solution of f in the interval $[-3, -2]$.

4a Find $\max_{a \le x \le b} |f(x)|$ for $f(x) = (2 - e^x + 2x)/3, x \in [0, 1]$.

We first find the critical points where the derivative is zero:

$$f'(x) = \frac{2 - e^x}{3} = 0$$
$$x = \ln(2).$$

Then by the extreme value theorem, $\max_{0 \le x \le 1} |f(x)|$ occurs at one of $x = 0, \ln(2), 1$. Substituting these values into $|f(x)|$ gives

$$|f(0)| = \frac{1}{3}, \quad |f(\ln(2))| \approx 0.46209812, \quad |f(1)| = \frac{4 - e}{3} \approx 0.427239391,$$

and so we see the maximum occurs at $x = \ln(2)$.

4c Find $\max_{a \le x \le b} |f(x)|$ for $f(x) = 2x\cos(2x) - (x - 2)^2, x \in [2, 4]$.

We first find the critical points where the derivative is zero:

$$f'(x) = 2\cos(2x) - 4x\sin(2x) - 2(x - 2) = 0$$
$$x \approx 3.13111.$$

Then by the extreme value theorem, $\max_{0 \le x \le 1} |f(x)|$ occurs at one of $x = 2, 3.13111, 4$. Substituting these values into $|f(x)|$ gives

$$|f(2)| \approx 2.61457448, \quad |f(3.13111)| \approx 4.98143396, \quad |f(4)| \approx 5.16400027,$$

and so we see the maximum occurs at $x = 4$.

8 Find the third Taylor polynomial $P_3(x)$ for the function $f(x) = \sqrt{x+1}$ about $x_0 = 0$. Approximate $\sqrt{0.5}$, $\sqrt{0.75}$, $\sqrt{1.25}$, and $\sqrt{1.5}$ using $P_3(x)$, and find the actual (absolute and relative) errors.

To find $P_3(x)$, we first find the values of the derivatives at $x_0 = 0$:

$$\begin{aligned}
f(0) &= \sqrt{0+1} = 1 \\
f'(0) &= \frac{1}{2}(0+1)^{-1/2} = \frac{1}{2} \\
f''(0) &= -\frac{1}{4}(0+1)^{-3/2} = -\frac{1}{4} \\
f'''(0) &= \frac{3}{8}(0+1)^{-5/2} = \frac{3}{8}.
\end{aligned}$$

Then

$$P_3(x) = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3.$$

x	-0.5	-0.25	0.25	0.5
$f(x)$	0.70711	0.86603	1.1180	1.2247
$P_3(x)$	0.71094	0.86621	1.1182	1.2266
Absolute Error	0.0038307	0.00018553	0.00013007	0.0018176
Relative Error	0.0054175	0.00021424	0.00011634	0.0014841

12a Find the third Taylor polynomial $P_3(x)$ for $f(x) = 2x\cos(2x) - (x-2)^2$ and $x_0 = 0$, and use it to approximate $f(0.4)$.

To find $P_3(x)$, we first find the values of the derivatives at $x_0 = 0$:

$$\begin{aligned}
f(0) &= 2(0)\cos(0) - (0-2)^2 = -4 \\
f'(0) &= 2\cos(0) - 4(0)\sin(0) - 2(0-2) = 6 \\
f''(0) &= -8\sin(0) - 8(0)\cos(0) - 2 = -2 \\
f'''(0) &= -24\cos(0) + 16(0)\sin(0) = -24.
\end{aligned}$$

Then

$$P_3(x) = -4 + 6x - x^2 - 4x^3$$

and

$$f(0.4) \approx P_3(0.4) = -2.016.$$

12b Use the error formula in Taylor's theorem to find an upper bound for the absolute error $|f(0.4) - P_3(0.4)|$. Compute the actual absolute error.

The error formula gives

$$R_3(x) = \frac{64\sin(2\xi) + 32\xi\cos(2\xi)}{24}x^4,$$

and so

$$\begin{aligned}|R_3(0.4)| &= \left|\frac{64\sin(2\xi)+32\xi\cos(2\xi)}{24}0.4^4\right| \\ &\le \left|\frac{64+32(0.4)}{24}0.4^4\right| \\ &= 0.08192.\end{aligned}$$

The actual absolute error is

$$|f(0.4)-P_3(0.4)| = |-2.00263463+2.016| = 0.013365$$

which is within the predicted error bound.

24 The *error function* defined by

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt$$

gives the probability that any one of a series of trials will lie within x units of the mean, assuming that the trials have a normal distribution with mean 0 and standard deviation $\frac{\sqrt{2}}{2}$. This integral cannot be evaluated in terms of elementary functions, so an approximating technique must be used.

(a) Integrate the Maclaurin series for e^{-x^2} to show that

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k+1}}{(2k+1)k!}.$$

Since the Maclaurin series for e^x is

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots$$

we know that

$$\begin{aligned}e^{-x^2} &= 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6} + \cdots \\ &= \sum_{k=0}^{\infty}\frac{(-1)^k x^{2k}}{k!}.\end{aligned}$$

Integrating this sum gives

$$\begin{aligned}\operatorname{erf}(x) &= \frac{2}{\sqrt{\pi}}\int_0^x\sum_{k=0}^{\infty}\frac{(-1)^k t^{2k}}{k!}\,dt \\ &= \frac{2}{\sqrt{\pi}}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\int_0^x t^{2k}\,dt \\ &= \frac{2}{\sqrt{\pi}}\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k+1}}{k!(2k+1)}.\end{aligned}$$

(b) The error function can also be expressed in the form

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} e^{-x^2} \sum_{k=0}^{\infty} \frac{2^k x^{2k+1}}{1 \cdot 3 \cdot 5 \cdots (2k+1)}.$$

Verify that the two series agree for $k = 1, 2, 3$, and 4. (Hint: Use the Maclaurin series for e^{-x^2}).

Substituting the Maclaurin series for e^{-x^2} into the above expression, we get

$$\begin{aligned}
&\frac{2}{\sqrt{\pi}} e^{-x^2} \sum_{k=0}^{\infty} \frac{2^k x^{2k+1}}{1 \cdot 3 \cdot 5 \cdots (2k+1)} \\
&\quad = \frac{2}{\sqrt{\pi}} \left[1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6} + \frac{x^8}{24} + \cdots\right] \\
&\qquad\qquad \cdot \left[x + \frac{2x^3}{3} + \frac{4x^5}{15} + \frac{8x^7}{105} + \frac{16x^9}{945} + \cdots\right] \\
&\quad = \frac{2}{\sqrt{\pi}} \left[x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{42} + \frac{x^9}{216} + \cdots\right] \\
&\quad = \frac{2}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{k!(2k+1)} = \operatorname{erf}(x).
\end{aligned}$$

(c) Use the series in part (a) to approximate erf(1) to within 10^{-7}.

Summing 11 terms in the series gives us $\operatorname{erf}(1) \approx 0.842700794090834$.

(d) Use the same number of terms as in part (c) to approximate erf(1) with the series in part (b).

Summing 11 terms in the second series gives $\operatorname{erf}(1) \approx 0.842700790029219$.

(e) Explain why difficulties occur using the series in part (b) to approximate erf(1).

The first series is alternating, so we may apply the alternating series test to get an idea of how far off our approximation is. The second is not alternating, so we have no such guide.