Math 104A - Midterm

Name:
Perm#:

Directions: You must show all your work to receive full credit. If you need extra room, you may use the back of the page, but clearly mark which problem you are solving.

1(a) *5 pts* State the intermediate value theorem.

If $f \in C[a,b]$, then for every K between $f(a)$ and $f(b)$, there is some $c \in [a,b]$ such that $f(c) = K$.

1(b) *5 pts* Use the intermediate value theorem to show that $x^5 - x^4 + x^3 - x^2 + x - 2 = 0$ has at least one solution.

Since $f(0) = -2$ and $f(2) = 32 - 16 + 8 - 4 + 2 - 2 = 20$, then by the intermediate value theorem, we know that there is some $c \in [0,2]$ such that $f(c) = 0$.

1(c) *5 pts* Using your results from part (b), perform two iterations of the bisection method.

i	a_i	b_i	p_i	$f(p_i)$
0	0	2	1	-1
1	1	2	1.5	3.15625
2	1	1.5	1.25	

1(d) *5 pts* How many iterations of the bisection method would be required to guarantee an approximation that is within 10^{-4} of the true solution?

The error bound at each step of the bisection method is given by $|p_n - p| \leq \frac{b-a}{2^n}$. Thus, in order for the error to be less than 10^{-4}, we would need

$$\begin{aligned} \frac{2-0}{2^n} &\leq 10^{-4} \\ 2 \cdot 10^4 &\leq 2^n \\ n\ln(2) &\geq \ln(2 \cdot 10^4) \\ n &\geq \frac{\ln(2 \cdot 10^4)}{\ln(2)} \approx 14.2877124. \end{aligned}$$

So, at least 15 iterations.

2(a) *10 pts* State Taylor's theorem.

If $f \in C^n[a,b]$ and $f^{(n+1)}(x)$ exists on (a,b), then for every $x, x_0 \in [a,b]$, we can write $f(x) = P_n(x) + R_n(x)$, where

$$\begin{aligned} P_n(x) &= f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 \\ &\quad + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n \\ &= \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k \end{aligned}$$

and

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)^{n+1}$$

for some ξ between x and x_0.

2(b) *5 pts* Compute the fourth Taylor polynomial of $f(x) = \frac{e^x + e^{-x}}{2}$ around the point $x_0 = 0$ evaluated at x.

First we compute derivatives evaluated at $x_0 = 0$:

$$\begin{aligned} f(0) &= 1 \\ f'(0) &= \frac{e^0 - e^{-0}}{2} = 0 \\ f''(0) &= \frac{e^0 + e^{-0}}{2} = 1 \\ f^{(3)}(0) &= \frac{e^0 - e^{-0}}{2} = 0 \\ f^{(4)}(0) &= \frac{e^0 + e^{-0}}{2} = 1. \end{aligned}$$

Then

$$P_4(x) = 1 + \frac{1}{2}x^2 + \frac{1}{24}x^4.$$

2(c) *5 pts* Give a bound for the error you'd commit by using your answer to part (b) to approximate $f(0.1)$.

The error is bounded by the remainder term

$$R_4(x) = \frac{f^{(5)}(\xi)}{120}x^5 = \frac{e^\xi - e^{-\xi}}{240}x^5,$$

where $\xi \in (0, 0.1)$. Since $e^\xi - e^{-\xi}$ is increasing, it is bounded by $e^{0.1} - e^{-0.1} \approx 0.2003335$, and so

$$|P_4(0.1) - f(0.1)| = |R_4(0.1)| = \left| \frac{e^\xi - e^{-\xi}}{240}(0.1)^5 \right| \leq \frac{0.2003335}{240} \cdot 10^{-5} \approx 8.3472 \cdot 10^{-9}$$

3(a) *5 pts* Suppose you want to find a zero p of a function $f(x)$. Newton's method is given by $p_{n+1} = g(p_n)$, where $g(x)$ is given by:

$$g(x) = x - \frac{f(x)}{f'(x)}$$

3(b) *5 pts* Write g as its first (linear) Taylor polynomial plus the remainder term around the point $x_0 = p$, evaluated at $x = p_n$.

First, using the fact that $f(p) = 0$, we have that

$$\begin{aligned} g(p) &= p - \frac{f(p)}{f'(p)} = p, \\ g'(p) &= 1 - \frac{f'(p)^2 - f(p)f''(p)}{f'(p)^2} = \frac{f(p)f''(p)}{f'(p)^2} = 0, \\ g''(p) &= \frac{[f'(p)f''(p) + f(p)f'''(p)]f'(p)^2 - 2f(p)f'(p)f''(p)^2}{f'(p)^4} = \frac{f''(p)}{f'(p)}. \end{aligned}$$

Then

$$\begin{aligned} p_{n+1} = g(p_n) &= g(p) + g'(p)(p_n - p) + \frac{g''(\xi_n)}{2}(p_n - p)^2 \\ &= p + \frac{g''(\xi_n)}{2}(p_n - p)^2, \end{aligned}$$

for some ξ_n between p and p_n.

3(c) *5 pts* Using the fact that $f(p) = 0$, show that Newton's method converges quadratically, assuming $f'(p) \neq 0$.

By the previous problem,

$$|p_{n+1} - p| = \left| \frac{g''(\xi_n)}{2}(p_n - p)^2 \right| = \left| \frac{g''(\xi_n)}{2} \right| |p_n - p|^2.$$

Then

$$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|^2} = \lim_{n\to\infty} \left| \frac{g''(\xi_n)}{2} \right| \frac{|p_n - p|^2}{|p_n - p|^2} = \left| \frac{g''(p)}{2} \right| = \left| \frac{f''(p)}{2f'(p)} \right| < \infty,$$

and so Newton's method converges quadratically.

4(a) *5 pts* Construct the linear Lagrange interpolating polynomial which interpolates the data $f(0) = 0$ and $f(1) = 1.1752$, and use it to approximate $f(0.5)$.

The interpolating polynomial is given by

$$\begin{aligned} P_{0,1}(x) &= f(x_0)\frac{x - x_1}{x_0 - x_1} + f(x_1)\frac{x - x_0}{x_1 - x_0} \\ &= 0 \cdot \frac{x - 1}{0 - 1} + 1.1752 \cdot \frac{x - 0}{1 - 0} \\ &= 1.1752x. \end{aligned}$$

Then $f(0.5) \approx P_{0,1}(0.5) = 0.5876$

4(b) *5 pts* Using Neville's iterated interpolation, find $P_{0,1,2}(0.5)$ using the following data:

j	x_j	$f(x_j)$
0	0	0
1	1	1.1752
2	2	3.6269

$$Q_{2,2} = \frac{(0.5-0)\cdot 1.1752 - (0.5-1)\cdot 0}{1-0} = 0.5876$$
$$Q_{3,2} = \frac{(0.5-1)\cdot 3.6269 - (0.5-2)\cdot 1.1752}{2-1} = -0.05065$$
$$Q_{3,3} = \frac{(0.5-0)\cdot(-0.05065) - (0.5-2)\cdot 0.5876}{2-0} = 0.4280375$$

In table form,

0	0		
1	1.1752	0.5876	
2	3.6269	-0.05065	0.4280375

So $P_{0,1,2}(0.5) = 0.4280375$.

4(c) *5 pts* The data from the previous problems were given by $f(x) = \frac{e^x - e^{-x}}{2}$. Compute the relative and absolute errors from part (a) and part (b).

First, the actual value is $f(0.5) = 0.521095305$. From part (a),

$$\text{absolute error: } |0.521095305 - 0.5876| = 0.066504695$$
$$\text{relative error: } \frac{|0.521095305 - 0.5876|}{|0.521095305|} = 0.127624821$$

From part (b),

$$\text{absolute error: } |0.521095305 - 0.4280375| = 0.093057805$$
$$\text{relative error: } \frac{|0.521095305 - 0.4280375|}{|0.521095305|} = 0.178581162$$