Math 5A - Midterm - 7/16

Name: ______________________________ Perm: ______________

- Read all directions carefully.
- Show all your work. Problems without work shown will receive no credit.
- No calculators or notecards.
- Good luck!

Problem 1 (*10 points*): Frank the mad scientist is in his lab, working on his Doomsday Device$^{\text{TM}}$. It just so happens that the device relies on a precisely calibrated mass-spring system. Frank determines that the system requires a mass of 1 kilogram, a spring with spring constant 1 $\frac{\text{Newtons}}{\text{meter}}$, and a dashpot which produces a damping constant of 2 $\frac{\text{Newtons}\cdot\text{second}}{\text{meter}}$.

In order to start the device, the mass is stretched 0.5 meters beyond its resting length, and let go without imparting any initial velocity. There are no external forces acting on the mass. Help Frank complete the plans for his Doomsday Device$^{\text{TM}}$ by determining the position of the mass as a function of time.

- 1 point for writing down the characteristic equation.
- 1 point for finding the correct characteristic root.
- 2 points for writing down the correct general solution.
- 2 points for using the initial conditions to get two equations.
- 1 point for solving for the constants in the general solution correctly.
- 3 points for writing the exact solution down.

Problem 2(a) (*5 points*): Frank's lab assistant, Joe, decided to "help" Frank by putting in a much more "efficient" spring into the Doomsday Device$^{\text{TM}}$. The new spring has a spring constant of $2\frac{\text{N}}{\text{m}}$. What is the general solution to the motion of the new and "improved" mass-spring system?

- 1 point for writing down the characteristic equation.
- 1 point for finding the correct characteristic roots.
- 3 points for writing down the correct general solution.

Problem 2(b) (*5 points*): After the device malfunctioned (and after he fired Joe), Frank discovers that the spring had fused with the device, making it unable to be replaced. Frank brilliantly determines that the problem can be fixed by applying a periodic magnetic field to the mass. The resulting external forcing function is $f(t) = \cos(2t)$. What is the general solution to the motion of the corrected mass-spring system?

- 1 point for writing down the correct guess for using undetermined coefficients.
- 1 point for substituting the guess correctly into the differential equation.
- 2 points for equating coefficients and solving for the constants correctly.
- 1 point for writing down the general solution.

Problem 3 (*10 points*): While Frank is merrily wreaking havoc in the local town using his Doomsday Device$^{\text{TM}}$, he notices that the mass-spring system settles into a periodic motion after a long time. Frank determines that this is from the steady-state solution to the motion of the mass-spring system. What is the steady state solution? Express your answer in phase-amplitude form.

- 2 points for identifying the steady state solution.
- 2 points for the correct definition of the amplitude and calculation of the amplitude.
- 2 points for the correct definition of the phase angle and its calculation.
- 4 points for writing down the steady state in phase amplitude form. This included writing down the general form, $A\cos(\omega t - \delta)$.

Problem 4 (*10 points*): Before Frank could do very much damage, his arch-nemesis Resonance Man determined the resonant frequency of the Doomsday Device$^{\text{TM}}$ and destroyed it. Frank was thrown into a high-security jail, but as he was thrown into his cell, he noticed the guard using a keypad next to the door. It looks like this:

$$A = \begin{bmatrix} 2 & 1 & 2 \\ 2 & 2 & 3 \\ 4 & 3 & 5 \end{bmatrix}.$$

Frank determines that if he can calculate the image of the linear transformation $T(\mathbf{v}) = A\mathbf{v}$, he can short-circuit the locking mechanism. Help Frank escape prison by calculating the basis for $\mathrm{Im}(T)$.

- 2 points for writing down anywhere that the image is the span of the column vectors.
- 5 points for row reducing and identifying pivot columns correctly.
- 3 points for writing down the correct basis.

Problem 5(a) (*8 points*): Frank made a mistake in his calculations, and realizes that he actually needs to determine the *kernel* of T. Help Frank escape prison once and for all by calculating the basis for $\text{Ker}(T)$.

- 3 points for row reducing (if already done in #4, then that counts too).
- 3 points for writing down the correct solution to the equation $A\mathbf{v} = \mathbf{0}$.
- 2 points for writing down the correct basis.

Problem 5(b) (*2 points*): What is the nullity of T? What is the rank of T?

- 1 point for the correct rank.
- 1 point for the correct nullity.